# On Cost of AND-EXOR Expansion of Switching Functions with Non-Equivalent Partial Symmetry

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*Abstract*— ***We consider the problem of cost evaluation for AND-EXOR representation of a switching function partially (non-equivalent) symmetric in $k$ variables. The evaluated parameters are the number of distinct polarities, the number of distinct coefficients and their weights (the number of products and literals assigned to the coefficient). The method has been verified by experiments with standard benchmark function having partial non-equivalent symmetries.***

**Index Terms.** *Switching functions, partial and total symmetry, AND-EXOR representation, optimal polarity*

## I. Introduction

Detection of the symmetry in a given function is a prerequisite to many methods of computer aided design of integrated circuits, for example, Boolean matching [10], [12], efficient determining variable orders in Binary Decision Diagrams (BDD) [2], circuits design and decomposition [7] and so on. Among different approaches to detect and exploit the symmetries, AND-EXOR expansion of symmetric functions are of special interest [9], [10], [11], due to the fact that AND-EXOR representation of some switching functions is more compact, testable and and possess many other advantages.

A technique to detect symmetries of switching functions in AND/EXOR forms has been suggested in [6]. Paper [9] was devoted to constructing a FPRM form for totally symmetric function based on a rectangular binary table and finding a minimal FPRM form from it. Authors of [10], [11] considered detecting of equivalent and non-equivalent symmetries through FPRM forms. None of these approaches exploits the properties of optimal polarities of AND-EXOR of symmetric switching function, i.e. quasi optimal polarity was chosen as the result of minimization strategy. Papers [5], [1] contain the complexity estimations for a particular case - totally symmetric functions. In [3] a polynomial time algorithm has been proposed for minimization of AND-EXOR expressions for totally symmetric switching functions. The problem to find an optimal polarity even for more simple case of totally symmetric function and especially for partially symmetric ones is not solved yet.

[†] *Support in part from the St. Francis Xavier University (Dept. of Computer Science), Antigonish, Nova Scotia (CANADA) and from the Polish State Committee for Scientific Research (POLAND) is acknowledged*

[‡] *The author gratefully acknowledges the facilities provided as James Chair Professor from the St. Francis Xavier University, Antigonish, Nova Scotia (CANADA)*

Motivated by these problems and applications, we evaluate, in a formal way, an optimal AND-EXOR representation for switching functions with non-equivalent partial symmetries of varibles. The evaluated characteristics are the number of products and literals for each AND-EXOR forms, or Reed-Muller forms of different polarities (the polarity of variable indicates that the variable is complemented or not). We are based on the previoys results obtained in more general case - for partially symmetric multiple-valued functions [8]. Our evaluation is exact and has been justified by a numerous experimental data given in the sections below.

## II. Non-equivalent symmetry of variables

Let $f$ be a switching function specified on a set of variables $X = \{x_1, x_2, \dots, x_n\}$. $f$ is *partially* symmetric with respect to $X_k \subseteq X$ if any permutation of variables in $X_k$ leaves $f$ unchanged [4]. In [10] such type of symmetry is refereed as non-equivalent.

*Example 1:* Function $f = x_1 \vee x_2x_3$ is a partially non-equivalent symmetric in variables $\{x_2, x_3\}$; function $f = x_1\overline{x}_3\overline{x}_4 \vee x_1x_3x_4 \vee \overline{x}_1x_2\overline{x}_5 \vee \overline{x}_1\overline{x}_2x_5$ is non-equivalent symmetric with respect to sets of variables $\{x_3, x_4\}$ and $\{x_2, x_5\}$, function $f = x_1x_2 \vee x_2x_3 \vee x_1x_3$ is totally non-equivalent symmetric.

The length of carrier vector for a function which is partially symmetric in $k$ variables and has no symmetries with respect to remaining $n-k$ variables, is $(k+1)2^{n-k}$ accordingly. The number of *distinct* positive polarity AND-EXOR coefficients for such function takes the same values, namely, $(k+1)2^{n-k}$.

In [8] a *symmetry vector* of a function $f$, $\mathbf{S} = [s_1s_2...s_n]$, where $s_i = s_j = 1$ iff $f$ is unchanged by an interchange of variables $x_i$ and $x_j$, has been introduced.

*Example 2:* The symmetry vector $\mathbf{S} = [1011]$ specifies a function that is partially non-equivalent symmetric in variables $\{x_1, x_3, x_4\}$.

## III. AND-EXOR expansion for symmetric functions

AND-EXOR, or fixed polarity $c$, $c = 0, 1, \dots, 2^n - 1$, Reed-Muller, expression for a switching function of $n$ vari-

TABLE I

THE NUMBER OF DISTINCT AND-EXOR EXPRESSIONS FOR A 20-VARIABLE SWITCHING FUNCTIONS SYMMETRIC IN $k$ VARIABLES

| | The number of symmetric variables, $k$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $n$ | 1 | 3 | 4 | 17 | 18 | 19 | 20 |
| 1 | 2 | | | | | | |
| 2 | 4 | 3 | | | | | |
| 3 | 8 | 6 | 4 | | | | |
| 4 | 16 | 12 | 8 | | | | |
| 5 | 32 | 24 | 16 | | | | |
| 6 | 64 | 48 | 32 | | | | |
| 7 | 128 | 96 | 64 | | | | |
| 8 | 256 | 192 | 128 | | | | |
| 9 | 512 | 384 | 256 | | | | |
| 10 | 1024 | 768 | 512 | | | | |
| 11 | 2048 | 1536 | 1024 | | | | |
| 12 | 4096 | 3072 | 2048 | | | | |
| 13 | 8192 | 6144 | 4096 | | | | |
| 14 | 16384 | 12288 | 8192 | | | | |
| 15 | 32768 | 24576 | 16384 | | | | |
| 16 | 65536 | 49152 | 32768 | | | | |
| 17 | 131072 | 98304 | 65536 | 18 | | | |
| 18 | 262144 | 196608 | 131072 | 36 | 19 | | |
| 19 | 524288 | 393216 | 262144 | 72 | 38 | 20 | |
| 20 | 1048576 | 786432 | 524288 | 144 | 76 | 40 | 21 |

ables is defined over GF(2) as follows [6]

$$R_c(X) = \sum_{j=0}^{2^n-1} r_c^{(j)} (x_1 \oplus c_1)^{j_1} (x_2 \oplus c_2)^{j_2} \ldots (x_n \oplus c_n)^{j_n},$$

$$(x_i + c_i)^{j_i} = \begin{cases} 1 & \text{if } j_i = 0, c_i = 0,1 \\ x_i & \text{if } j_i = 1, c_i = 0 \\ \overline{x}_i & \text{if } j_i = 1, c_i = 1 \end{cases}$$

where $r_c^{(j)}$ is $j$-th coefficient of AND-EXOR expansion, $j = 0, 1, ..., 2^n - 1$; $c_i$ is $i$-th digit of binary representation of polarity $c$, $i = 1, ..., n$, $j_i$ is $i$-th digit of binary representation $j_1 j_2 ... j_n$ of an assignment $j$ of values to variables $x_1, x_2, ..., x_n$.

All $2^n$ AND-EXOR expressions of a function partially symmetric with respect to variables $x_{t_q} \in X_k$, $q = 1, 2, ..., k$, can be grouped according to the number of variables whose form is $x_{t_q}, \overline{x}_{t_q}$.

*Definition 1: The distinct group* of AND-EXOR expressions of a function $f$ partially symmetric in $k$ variables $\{x_{t_1}, ..., x_{t_k}\}$ is a group of expressions of polarities $c = c_1 ... c_{t_1} ... c_{t_k} ... c_n$ such that to satisfy the linear equation

$$c_{t_1} + \ldots + c_{t_k} = C, \tag{1}$$

for $C = 0, 1, \ldots, k$.

For an $n$-variables switching function symmetric with respect to $k$ variables $\{x_{t_q}\}$, the number of distinct groups of AND-EXOR expressions is $N_d = (k+1)2^{n-k}$ [8].

Table I contains values of $N_d$ for different $k$ and $2 \leq n \leq 20$ for a switching function.

*Definition 2:* [8] The *symmetry distribution vector* is two elements vector $[k_0 k_1]$, where $k_0 \subseteq n_0$, $k_1 \subseteq n_1$, and $k_i$ is the number of $l$'s in the corresponding $k$ digits $c_{i_1}, c_{i_2}, ..., c_{i_k}$ of the code $c_1 c_2 ... c_n$ of polarity $c$, so that $k_0 + k_1 = k$.

*Example 3:* Given a 4-variable function symmetric with respect to variables $\{x_2, x_3, x_4\}$, $S = [0111]$, its AND-EXOR expression of polarity $c = 5$, i.e. $c_1 c_2 c_3 c_4 = 0101$. The symmetry distribution vector is $[k_0 k_1] = [12]$.

TABLE II

THE NUMBER OF DISTINCT COEFFICIENTS, $N_c$, IN AND-EXOR FORM OF POLARITY $p$, FOR A 10-VARIABLE SWITCHING FUNCTION SYMMETRIC IN $2 \leq k \leq 10$ VARIABLES

| $p$ | $k$=2 | $k$=3 | $k$=4 | $k$=5 | $k$=6 | $k$=7 | $k$=8 | $k$=9 | $k$=10 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 768 | 512 | 320 | 192 | 112 | 64 | 36 | 20 | 11 |
| 1 | 768 | 512 | 320 | 192 | 112 | 64 | 36 | 20 | 20 |
| 5 | 768 | 512 | 320 | 192 | 112 | 64 | 64 | 48 | 36 |
| 10 | 768 | 512 | 320 | 192 | 112 | 112 | 84 | 60 | 11 |
| 19 | 768 | 512 | 320 | 192 | 192 | 144 | 100 | 20 | |
| 35 | 768 | 512 | 320 | 320 | 240 | 160 | 36 | | |
| 63 | 768 | 512 | 320 | 320 | 256 | 64 | | | |
| 111 | 768 | 512 | 512 | 384 | 112 | | | | |
| 191 | 768 | 768 | 576 | 192 | | | | | |
| 319 | 1024 | 768 | 320 | | | | | | |
| 511 | 1024 | 512 | | | | | | | |
| 767 | 768 | | | | | | | | |

It has been proved in [8], the following has been proved: given the symmetry distribution vector $[k_0 k_1]$, the number of different $c$-polarity AND-EXOR expressions within one of $N_d$ distinct groups, is the multinomial of $k$ over the partition $k_0 + k_1 = k$

$$N_g = \frac{k!}{k_0! k_1!}.$$

A distinct AND-EXOR expression of a switching function partially symmetric in $k$ variables is characterized by

*(i) distinct groups of products (within all $2^n$ products) in the expression, each is assigned to a distinct coefficients,*

*(ii) the number of products within a distinct group of products* and

*(iii) the number of literals within all products of a distinct group of products.*

Each $c$-polarity AND-EXOR expression of a function is represented by its $2^n \times 1$ coefficients vector. For a function partially symmetric with respect to $k$ variables, there are some distinct groups of coefficients within the coefficients vector, and the number of the groups is characterized by the number $k$ and also the symmetry distribution vector $[k_0 k_1]$ for the given polarity $c$. It has been proved in [8] that the number $N_c$ of distinct coefficients in the corresponding $c$-polarity AND-EXOR expansion of a switching function partially symmetric with respect to $k$ variables, be

$$N_c = (k_0 + 1)(k_1 + 1)2^{n-k}.$$

The particular cases $c = 0$ and $c = 2^n$ correspond to the upper bound of the number of distinct coefficients of AND-EXOR expansion of a partially symmetric switching function. A partially symmetric function of $n$ variables can be represented with $(k+1)2^{n-k}$ distinct 0 (and also $(2^n)$)-polarity AND-EXOR coefficients.

Table II illustrates the dependence of the number of distinct coefficients, $N_c$, on the number $k$ of symmetric variables and some of distinct polarity $p$ for a 10-variable switching function.

If we know the number $N_c$ of coefficients (distinct groups of products) in a $c$-polarity form of a partially symmetric functions, as well as the cost of each coefficient (the number of products within the group), we can evaluate the product cost of the AND-EXOR expression. We also can find the number of literals within a product (so-called literal cost).

Then one can choose an expression with the lowest product and/or literal cost.

So, the optimality of an AND-EXOR expression is determined by its cost evaluated through: the number of non-zero products and the number of non-zero literals within the products.

Each of $N_c$ coefficients $r_c^{(p)}$, $p = 0, ..., N_c - 1$, of $c$-polarity AND-EXOR expression is assigned to a modulo-2 sum of products, i.e. a group of products $(x_1 \oplus c_1)^{j_n} ... (x_n \oplus c_n)^{j_n}$. Let us define $N_p$ as the number of products assigned to such distinct coefficient, and let $N_l$ be a number of literals within one product. Clear, that each products consists of $N_l = \sum_{i=1}^{n} j_i$ literals.

It has been proved in [8] that the product cost $N_p$ of the coefficient $r_c^{(p)}$ within $c$-polarity AND-EXOR expression of a function partially symmetric with respect to $k$ variables $x_{t_1}, ..., x_{t_k}$, i.e. the number of products within $p$-th modulo-2 sum of products, $p = 0, ..., N_c - 1$, is equal to

$$N_p(r_p^{(j)}) = \frac{k_1!}{k0_{C(A)}!k1_{C(A)}!} \frac{k_0!}{k0_{\overline{C}(A)}!k1_{\overline{C}(A)}!}.$$

*Example 4:* Let $f$ be a 4-variable function symmetric with respect to variables $\{x_2, x_3, x_4\}$, $k = 3$. Consider its AND-EXOR form when $c = 5$, $c_1c_2c_3c_4 = 0101$, $n_0 = 2$, $n_1 = 2$. The number $N_p$ of products of terms within all possible $N_c = (k_0 + 1)(k_1 + 1)2^{4-3} = 3 \cdot 2 \cdot 2 = 12$ groups of products in the 5-polarity AND-EXOR form are shown in Table III. Here $j_1j_2j_3j_4$ represent all combinations of variable values, $k0_{C(A)}$ and $k1_{C(A)}$ correspond to the number of 0's and 1's in digits $j_1, j_2, j_3$ (labeled as $g$) which corresponds to 1's in the code $c_1c_2 \ldots c_n$ of the polarity $c$.

Consider, for instance, the product cost of the coefficient $r_5^{(2)}$. Taking into account that $c_1c_2c_3c_4 = 0101$, the code $j = 1 = 0001$ is divided to the following parts: first, $j_2j_4 = 0\ 1$ (it corresponds to $c_2c_4 = 1\ 1$), $k_1 = 2$, $k0_{C(A)} = 1$, $k1_{C(A)} = 1$, and, second, $j_1j_3 = 0\ 1$ (it corresponds to $c_1c_3 = 0\ 0$), $k_0 = 1$, $k0_{\overline{C}(A)} = 1$, $k1_{\overline{C}(A)} = 0$. So, $(k_1!/k0_{C(A)}!k1_{C(A)}!)(k_0!/k0_{\overline{C}(A)}!k1_{\overline{C}(A)}!) = (2!/1!1!)(1!/0!1!) = 2$. These two terms are coded by $p_1p_2p_3p_4 = 0001, 0100$ and written as follows: $(x_1 + 0)^0(x_2 + 1)^0(x_3 + 0)^0(x_4 + 1)^1 = \overline{x}_4$ and $(x_1 + 0)^0(x_2 + 1)^1(x_3 + 0)^0(x_4 + 1)^0 = x_2$.

Each product, to which a distinct coefficient $r_p^{(j)}$ is assigned, contains some non-zero literals of the form $(x_i + p_i)$.

*Theorem 1:* The literal cost of the coefficient $r_p^{(j)}$ of the $p$-polarity AND-EXOR expansion for a $m$-valued logic function symmetric in $k$ variables is defined by the expression:

$$N_l(r_p^{(j)}) = N_p(r_p^{(j)}) \sum_{l=0}^{1} nl_q \tag{2}$$

*Proof:* To count the literal cost of the coefficient, we have to know the number of such literals, i.e. how many non-zero literals are included in such a product. For the product $(x_1 + p_1)(x_2 + p_2) \ldots (x_n + p_n)$, the number of literals with with $q$-tuple exponentiation $(x_i + p_i)^q$ is counted

| $p$ | $j_1j_2j_3j_4$ | $k0_0$ | $k1_0$ | $k0_1$ | $k1_1$ | $N_p$ |
|---|---|---|---|---|---|---|
| 0 | 0000 | 1 | 0 | 2 | 0 | 1 |
| 1 | 0001<br>0100 | 1 | 0 | 1 | 1 | 2 |
| 2 | 0010 | 0 | 1 | 2 | 0 | 1 |
| 3 | 0011<br>0110 | 0 | 1 | 1 | 1 | 2 |
| 4 | 0101 | 1 | 0 | 0 | 2 | 1 |
| 5 | 0111 | 0 | 1 | 0 | 2 | 1 |
| 6 | 1000 | 1 | 0 | 2 | 0 | 1 |
| 7 | 1001<br>1100 | 1 | 0 | 1 | 1 | 2 |
| 8 | 1010 | 0 | 1 | 2 | 0 | 1 |
| 9 | 1011<br>1110 | 0 | 1 | 1 | 1 | 2 |
| 10 | 1101 | 1 | 0 | 0 | 2 | 1 |
| 11 | 1111 | 0 | 1 | 0 | 2 | 1 |

TABLE III
THE PRODUCT COSTS $N_p$ OF COEFFICIENTS $r_5^{(p)}$ FOR AND-EXOR EXPRESSION OF POLARITY $c = 5$ ($c_1c_2c_3c_4 = 0101$) FOR A 4-VARIABLES FUNCTION PARTIALLY SYMMETRIC IN VARIABLES $\{x_2, x_3, x_4\}$, $k_1 = 1$, $k_0 = 2$

as the summarized number of 1's in the assignment $A$: $\sum_{l=0}^{1} nl_1$. The total number of literals, no matter to which exponentiation, be $\sum_{l=0}^{1} nl_1$. Hence, the total number of literals in all products assigned to the coefficient $r_p^{(j)}$, is counted by equation (2). ■

*Theorem 2:* The literal cost of the $c$-polarity AND-EXOR expression of an $m$-valued $n$-variable logic function partially symmetric in $k$ variables, is counted as

$$N_l(f_p) = \sum_{j=0}^{N_c-1} r_p^{(j)} N_l(r_p^{(j)}) \tag{3}$$

*Proof:* The proof follows from Theorem 1. ■

The sum of product costs $N_p$ of all products in a $c$-polarity AND-EXOR expression yields the total product cost of the expression.

*Theorem 3:* The product cost of the $c$-polarity AND-EXOR expression for a function partially symmetric with respect to $k$ variables is equal to

$$N_p(f) = \sum_{p=0}^{N_c-1} r_c^{(p)} \frac{k_1!}{k0_{C(A)}!k1_{C(A)}!} \frac{k_0!}{k0_{\overline{C}(A)}!k1_{\overline{C}(A)}!} \tag{4}$$

*Proof:* Proof follows immediately from Theorem ?? ■

The sum of literal costs $N_p$ of all products in a $c$-polarity AND-EXOR expression yields the total literal cost of the expression.

*Theorem 4:* The literal cost of the $c$-polarity AND-EXOR expression for a function partially symmetric with respect to $k$ variables is equal to

$$N_l(f) = \sum_{p=0}^{N_c-1} r_c^{(p)} N_p N_l \tag{5}$$

/* (input)
$f(x_1, \ldots, x_n) : \{0,1\}^n \rightarrow \{0,1\}$
$x_{i_1}, \ldots, x_{i_k}$ - symmetric variables */
/* (output)
$np$ - number of products of optimal AND-EXOR expression
$nl$ - number of literals of optimal AND-EXOR expression */
/* $\mathrm{N}_{prod}$ - vector with number of products of every distinct polarities
$\mathrm{N}_{lit}$ - vector with number of literals of every distinct polarities */
{
  $N_d \leftarrow (k+1) * 2^{n-k}$ /* evaluate number of distinct polarities */
  for($\forall i \in 0, 1, \ldots, N_d - 1$)
  {
    $j \leftarrow$ index[$i$] /* derive the index of next i-th group of distinct polarities */
    $F_c[i] \leftarrow$ coefficient_vector(c) /* evaluate the AND-EXOR vector of length $\mathrm{N}_c$ for the polarity c */
    $\mathrm{N}_t[i] \leftarrow$ weight_vector(c) /* derive the weight vector */
    $\mathrm{N}_{prod}[i] \leftarrow \sum_{p=0}^{N_c} N_k^{(p)}[i] * F_c^{(p)}[i]$ /* calculate the component-wise product and derive the cost of the AND-EXOR form as the sum of its coefficient */
    $\mathrm{N}_l[i] \leftarrow \sum_{p=0}^{N_c} j^p[i]$ /*Calculate the number of 1's in the code $j_1 \ldots j_n$*/
    $\mathrm{N}_{lit}[i] \leftarrow \sum_{p=0}^{N_c} N_t^{(p)}[i] * F_c^{(p)}[i] * N_l[i]$ /*Calculate the number of literals in AND-EXOR form from the coefficients vector, $N_t^{(p)}[i]$ is $p$-th element in $N_t[i]$ */
    }
  $l \leftarrow$ i : $\mathrm{N}_{prod}[i]$=min($\mathrm{N}_p[j], \forall j \in \{0, 1, \ldots, N_d - 1\}$)
  $np \leftarrow \mathrm{N}_{prod}[l]$
  $nl \leftarrow \mathrm{N}_{lit}[l]$
}

Fig. 1. Pseudo code of the algorithm to derive a minimal product and literal cost of a $c$-polarity AND-EXOR expression of a function partially symmetric in $k$ variables

where $N_l = \sum_{i=1}^{n} j_i$ be the number of literals in a product within a distinct group (that is also equal to $k1_{C(A)}$ for the group).

*Proof:* Proof follows from Theorem **??**. ∎

The estimation of product and literal costs obtained in the subsections before, can be evaluated for each distinct AND-EXOR expression given its coefficient vector $R_c$. To find a minimal expression we have to choose a minimal of the cost of these distinct expressions.

Fig. 1 contain the pseudo code of the algorithm to evaluate the costs of the $c$-polarity AND-EXOR expression of a partially symmetric function and choose the minimal product and literal costs that correspond to a minimal AND-EXOR expression.

TABLE IV
THE PRODUCT COST OF $j$-TH DISTINCT COEFFICIENT $r_p^{(j)}$ OF $p$-POLARITY AND-EXOR EXPRESSION OF A 10-VARIABLE SWITCHING FUNCTION SYMMETRIC IN 6 VARIABLES

| Index | Polarity 0 | 16 | 32 | 48 | 64 | 80 | 96 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 6 | 1 | 2 | 3 | 4 | 5 | 6 |
| 32 | 15 | 5 | 1 | 3 | 6 | 10 | 15 |
| 48 | 20 | 5 | 4 | 1 | 4 | 10 | 20 |
| 64 | 15 | 10 | 8 | 3 | 1 | 5 | 15 |
| 80 | 6 | 10 | 4 | 9 | 2 | 1 | 6 |
| 96 | 1 | 10 | 6 | 9 | 8 | 1 | 1 |
| 112 | | 10 | 12 | 3 | 12 | 5 | |
| 128 | | 5 | 6 | 3 | 8 | 10 | |
| 144 | | 5 | 4 | 9 | 2 | 10 | |
| 160 | | 1 | 8 | 9 | 1 | 5 | |
| 176 | | 1 | 4 | 3 | 4 | 1 | |
| 192 | | | 1 | 1 | 6 | | |
| 208 | | | 2 | 3 | 4 | | |
| 224 | | | 1 | 3 | 1 | | |
| 241 | | | | 1 | | | |

TABLE V
THE PRODUCT COST OF $j$-TH DISTINCT COEFFICIENT $r_p^{(j)}$ OF $p$-POLARITY AND-EXOR EXPRESSION OF A 10-VARIABLE SWITCHING FUNCTION SYMMETRIC IN 9 VARIABLES

| Index | Polarity 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 126 | 28 | 14 | 6 | 1 | 5 | 15 | 35 | 70 | 126 |
| 19 | 1 | 70 | 35 | 45 | 5 | 40 | 45 | 14 | 1 | 1 |
| 27 | | 28 | 70 | 60 | 40 | 30 | 3 | 42 | 70 | |
| 35 | | 1 | 21 | 45 | 60 | 6 | 60 | 7 | 1 | |
| 36 | | | 7 | 45 | 40 | 4 | 45 | 21 | | |
| 38 | | | 14 | 15 | 10 | 20 | 18 | 35 | | |
| 42 | | | 1 | 18 | 20 | 40 | 1 | 21 | | |
| 47 | | | 1 | 6 | 20 | 4 | 15 | 1 | | |
| 52 | | | | 3 | 4 | 10 | 6 | | | |
| 55 | | | | 1 | 6 | 10 | 1 | | | |
| 59 | | | | | 1 | 1 | | | | |

## IV. EXPERIMENTAL RESULTS

We implemented the algorithm described in the previous section in C++. All experiments have been conducted on a Pentium 100Mhz with 48 MBytes of the main memory.

We fix the number of symmetric variables and observe how the costs of AND-EXOR coefficients depends on the polarity.

Tables IV, V and VI and Fig. 2 illustrate the distribution of the product cost of each distinct coefficient in AND-EXOR form of a 10-variable switching function symmetric in 6, 9 and 10 (totally) variables correspondingly. Note, that the symmetry is considered with respect to first six (nine) of 10 variables. For symmetry in 6 variables, and polarity $p = 0$, the cost is distributed among its 112 distinct coefficients as below: the maximum cost (equal to 20) is concentrated to 48 - 63 rd coefficients . The same is true for the last, 95th, polarity.

Now fix the polarity and the number of symmetric variables (9 of 10 variables) and observe, how the position of 0 in the symmetry vector (the number of non-symmetric variable) influences on the product cost of each coefficient in AND-EXOR expression.

The product cost of $j$-th distinct coefficient $r_0^{(j)}$ of 1-polarity AND-EXOR expression of a 10-variable switching function partially symmetric in 9 variables, with regard to the position of non-symmetric variable (marked by 0 in the symmetry vector $S = [1..101...1]$) is shown in Table VII and Fig. 3.

The next goal of our experiments was to implement the

TABLE VI

The product cost of $j$-th distinct coefficient $r_p^{(j)}$ of $p$-polarity AND-EXOR expression of a 10-variable switching function symmetric in 10 variables

| Index | Polarity | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 252 | 36 | 8 | 21 | 6 | 1 | 6 | 21 | 56 | 126 | 252 |
| 10 | 1 | 126 | 112 | 63 | 15 | 25 | 80 | 63 | 16 | 1 | 1 |
| 12 | | 84 | 70 | 35 | 90 | 10 | 24 | 105 | 112 | 36 | |
| 14 | | 36 | 70 | 105 | 15 | 100 | 6 | 21 | 112 | 126 | |
| 19 | | 1 | 56 | 35 | 20 | 50 | 36 | 105 | 8 | 1 | |
| 21 | | | 8 | 63 | 60 | 100 | 4 | 63 | 56 | | |
| 26 | | | 1 | 21 | 24 | 50 | 24 | 21 | 1 | | |
| 27 | | | | 7 | 36 | 50 | 4 | 35 | | | |
| 31 | | | | 1 | 4 | 5 | 20 | 1 | | | |
| 34 | | | | | 1 | 5 | 1 | | | | |
| 35 | | | | | | 1 | | | | | |

Fig. 2. The product cost of $j$-th distinct coefficient $r_p^{(j)}$ of $p$-polarity AND-EXOR expression of a 10-variable switching function symmetric in 9 variables (upper) and 10 variables (lower)

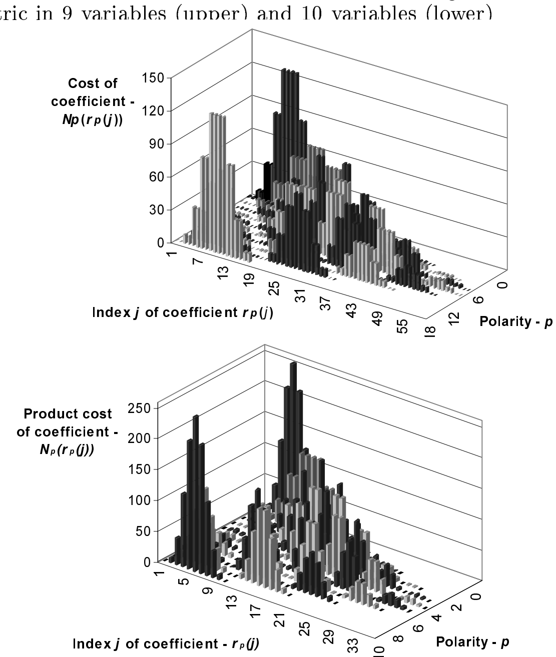
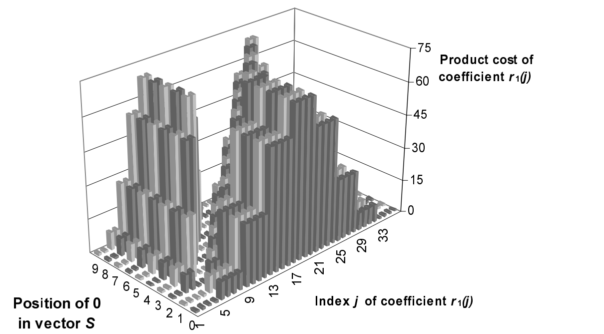



TABLE VII

The product cost of $j$-th distinct coefficient $r_p^{(j)}$ of 1-polarity AND-EXOR expression of a 10-variable switching function symmetric in 9 variables, $S = [1..101...1]$

| Index | Position of 0 in vector $S$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| 4 | 8 | 8 | 1 | 28 | 28 | 28 | 28 | 28 | 28 | 28 |
| 6 | 8 | 8 | 8 | 1 | 56 | 56 | 56 | 56 | 56 | 56 |
| 8 | 28 | 28 | 28 | 8 | 1 | 70 | 70 | 70 | 70 | 70 |
| 10 | 28 | 28 | 28 | 28 | 8 | 1 | 56 | 56 | 56 | 56 |
| 13 | 56 | 56 | 56 | 56 | 28 | 8 | 1 | 28 | 28 | 28 |
| 14 | 56 | 56 | 56 | 56 | 56 | 28 | 8 | 1 | 8 | 8 |
| 16 | 70 | 70 | 70 | 70 | 70 | 56 | 28 | 8 | 1 | 1 |
| 18 | 70 | 70 | 70 | 70 | 70 | 70 | 56 | 28 | 8 | 1 |
| 20 | 56 | 56 | 56 | 56 | 56 | 56 | 70 | 56 | 28 | 8 |
| 22 | 56 | 56 | 56 | 56 | 56 | 56 | 56 | 70 | 56 | 28 |
| 24 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 56 | 70 | 56 |
| 26 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 28 | 56 | 70 |
| 28 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 28 | 56 |
| 30 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 28 |
| 32 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 |
| 34 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Fig. 3. The product cost of $j$-th distinct coefficient $r_p^{(j)}$ of 1-polarity AND-EXOR expression of a 10-variable switching function sym-




proposed method to evaluate the weights of the distinct AND-EXOR expressions for benchmark functions known as partially non-equivalent symmetric with respect to given variables.

Our program has been implemented in $C^{++}$ and yields the number of products $N_p$ and literals $N_l$ within $N_d$ distinct polarities for symmetric functions.

The scheme of our experiments was as follows. First of all we recognized symmetries (partial and total) in the group of MCNC with 4-15 inputs benchmarks. For this purpose we used special recognition program [1].

In Tables VIII - IX we give fragment of our experimental study where $In$ is the number of variables; $P/L$ is the number of products/literals in EXOR expression; $t$−run time of minimization in seconds (Pentium 200MMX processor). Note that in this table in column *Exact optimum* we place the distinct polarities, and the number in the brackets corresponds to the total number of optimal polarities.

In our first experiment we have studied more simple case of partially symmetry, i.e. one block of symmetric variables (Table VIII). Note that in this table in column *Exact optimum* we place the numbers of distinct polarities, and the number in the brackets corresponds to the total number of optimal polarities.

The case of so called multi partial symmetry (several groups of variables in which a function is partially symmetric) was the subject of our second experiment. Table IX contains the fragment of results of recognition partially symmetries in MCNC benchmarks 4-15 inputs variables. Note that in this table in column *Exact optimum* we place the distinct polarities, and the number in the brackets corresponds to the total number of optimal polarities.

For instance, the result of AND-EXOR minimization function $bw11$ is $\overline{x}_0\overline{x}_2x_3\overline{x}_4 \oplus \overline{x}_0x_1\overline{x}_2\overline{x}_4$ for polarity $c = 21(c_1c_2c_3c_4c_5 = 10101)$ and $\overline{x}_0\overline{x}_2\overline{x}_3\overline{x}_4 \oplus \overline{x}_0\overline{x}_1\overline{x}_2\overline{x}_4$ for polarity $c = 31(c_1c_2c_3c_4c_5 = 11111)$.

## V. Concluding remarks

In the paper, we evaluated the cost of AND-EXOR expansion for switching functions with non-equivalent partial symmetries in variables. We found an optimal polarity forms (with minimal product and literal cost) for the

TABLE VIII

PARTIAL SYMMETRY: EXACT ESTIMATION OF OPTIMAL POLARITY OF AND-EXOR EXPRESSION OF SWITCHING FUNCTIONS

| Test example | | | Exact optimum | | |
|---|---|---|---|---|---|
| **Name** | **In** | **Sym.** | $c$ | $P/L$ | $t$ |
| bw3 | 5 | $x_1, x_4$ | 31 | 6/21 | 0.00 |
| 5x01 | 7 | $x_3, x_4$ | 69 | 12/44 | 0.28 |
| 5x6 | 7 | $x_5, x_6, x_7$ | 112,113,115,119* | 4/6 | 0.17 |
| sao24 | 10 | $x_6, x_{10}$ | 187 | 55/376 | 129.93 |
| 5x7 | 7 | $x_1, x_5 - x_7$ | 16,17,19,23,87(12) | 2/3 | 0.11 |
| f53 | 8 | $x_1, x_2$ | 0,64,192 | 11/32 | 1.96 |
| f55 | 8 | $x_1 - x_4$ | 14(16) | 4/6 | 0.79 |
| f56 | 8 | $x_1 - x_5$ | 2,10,26,58,122,250(33) | 2/3 | 0.49 |
| sao21 | 10 | $x_6, x_{10}$ | 819,825,921 | 36/248 | 129.93 |
| sao22 | 10 | $x_6, x_{10}$ | 155,179 | 52/374 | 129.93 |

TABLE IX

MULTI PARTIAL SYMMETRIES: COMPARISON EXACT ESTIMATION OF OPTIMAL POLARITY OF AND-EXOR EXPRESSION AND RESULTS OF HEURISTIC MINIMIZATION; THE NUMBER WITHIN THE BRACKETS MEANS HOW MANY OTHER BEST SOLUTION HAVE BEEN OBTAINED

| Test example | | | Exact optimum | | |
|---|---|---|---|---|---|
| **Name** | **In** | **Sym.** | $c$ | $P/L$ | $t$ |
| bw11 | 5 | $x_1, x_3, x_5;$ | 31 | 2/8 | 0.01 |
| | | $x_2, x_4$ | 21 | 2/8 | 0.01 |
| 5x10 | 7 | $x_1, x_7;$ | 65 | 3/13 | 0.27 |
| | | $x_2 - x_4;$ | 65 | 3/13 | 0.27 |
| | | $x_5, x_6$ | 65 | 3/13 | 0.27 |
| z41 | 7 | $x_1, x_4, x_7;$ | 0 | 15/56 | 0.17 |
| | | $x_2, x_5;$ | 0 | 15/56 | 0.27 |
| | | $x_3, x_6$ | 0 | 15/56 | 0.27 |
| z42 | 7 | $x_1, x_4, x_7;$ | 0,36,95,123 | 9/22 | 0.27 |
| | | $x_2, x_5;$ | 0,36,95 | 9/22 | 0.18 |
| | | $x_3, x_6$ | 0,36,95,123 | 9/22 | 0.27 |
| z43 | 7 | $x_1, x_4, x_7;$ | 0(16) | 5/8 | 0.18 |
| | | $x_2, x_5;$ | 0(8) | 5/8 | 0.18 |
| | | $x_3, x_6$ | 0(8) | 5/8 | 0.27 |
| z44 | 7 | $x_1, x_4, x_7;$ | 0(16) | 3/3 | 0.17 |
| | | $x_2, x_3, x_5, x_6$ | 0(8) | 3/3 | 0.11 |
| 5x5 | 7 | $x_1, x_4;$ | 33(4) | 7/15 | 0.27 |
| | | $x_5, x_6$ | 35(3) | 7/15 | 0.27 |
| f54 | 8 | $x_1, x_2, x_3;$ | 20(8) | 7/15 | 2.45 |
| | | $x_5, x_8$ | 20 | 7/15 | 2.40 |
| f57 | 8 | $x_1 - x_6;$ | 0(128) | 2/2 | 0.46 |
| | | $x_7, x_8$ | 0(128) | 2/2 | 1.96 |
| newtag | 8 | $x_1, x_3;$ | 160 | 6/28 | 3.80 |
| | | $x_5, x_6;$ | 160 | 6/28 | |
| | | $x_7, x_8$ | 160 | 6/28 | |
| m1811 | 15 | $x_7 - x_{15}$ | 20479 | 13/90 | 3596.65 |

benchmark functions, and show the experimental results. Our next goal is to improve the method to enlarge the size of the decided problem (the number of inputs in functions).

## REFERENCES


[1] Shmerko V. Butler J., Dueck G. and Yanushkevich S. On recognition of symmetric switching functions in reed-muller form. In *Proc. IAPR Int. Conf. on Pattern Recognition & Information Processing , Minsk, Belarus*, pages 215–234, 1999.

[2] J. Molitor D. Moller and R. Drechsler. Symmetry Based Variable Ordering for ROBDDs. In *Proc. IFIP Workshop on Logic and Architecture Synthesis, Grenoble*, pages 47–53, 1994.

[3] R. Drechsler and B. Becker. Sym*pathy*: Fast Exact Minimization of Fixed Polarity Reed-Muller expansion for symmetric functions. *IEEE Trans. on Computer Aided Design of Integrated Circuits and Systems*, 16(1):1–5, Jan. 1997.

[4] M. Harrison. *Introduction to Switching and Automata Theory.* McGraw-Hill, 1965.

[5] J. L. Nowlin J. T. Butler and T. Sasao. Planarity in ROMDD's of multiple- valued symmetric functions. In *Proc. of the Int. Symp. on Multiple-Valued Logic*, pages 83–88, May, 1996.

[6] P. Deschamps M. J. Davio and A. Thayse. *Discrete and Switching Functions.* McGraw-Hill Int. Book Co, 1978.

[7] I. Pomeranz and S. M. Reddy. On Determining Symmetries in Inputs of Logic Circuits. *Computer - Aided Design of Integrated Circuits and Systems*, 13(11):1478–1433, Nov. 1994.

[8] G. Dueck S. Yanushkevich, J. Butler and V. Shmerko. Experiments on fprm expressions for partially symmetric logic functions. In *Proc. of the Int. Symp. on Multiple-Valued Logic*, pages 141–146, May 2000.

[9] V. P. Suprun. Fixed Polarity Reed-Muller Expressions of Symmetric Boolean Functions. In *Proc. IFIP WG 10.5 Workshop on Application of the Reed-Muller Expansions in Circuit Design, Japan*, pages 246–249, 1995.

[10] Chien-Chang Tsai and M. Marek-Sadowska. Generalized Reed-Muller Forms as a Tool to Detect Symmetries. *IEEE Trans. on Computers*, C-45(1):33–40, Jan. 1996.

[11] Chien-Chang Tsai and M. Marek-Sadowska. Boolean Functions Classification via Fixed Polarity Reed-Muller Forms. *IEEE Trans. on Computers*, C-46(2):173–186, Feb. 1997.

[12] Kuo-Hua Wang and Ting Ting Hwang. Boolean Matching for Incompletely Specified Functions. *Computer - Aided Design of Integrated Circuits and Systems*, 16(2):160–168, Feb. 1997.