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Arithmetical Canonical Expansion of Boolean and MVL Functions as Generalized Reed-Muller Series

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Abstract - In the paper analogs of Reed-Muller canonical expansion, Boolean Differences and Taylor logic series are introduced for the case of arithmetical logic. Efficient matrix algorithms to synthesize the arithmetical forms for Boolean and multivalued (MVL) functions are given.

I. INTRODUCTION

Arithmetical logic is one of intensively developed branches of algebra of logic. Apparently, the first attempts to present logic operations by arithmetical ones were taken by the founder of Boolean algebra J.Boole. It is obvious that for Boolean variables x_1 and x_2 the following is true: $x_1 \vee x_2 = x_1 + x_2 - x_1 x_2$, $\overline{x_1} = 1 - x_1$, $x_1 \wedge x_2 = x_1 x_2$.

The elements of arithmetical logic have been applied in Artificial Intelligence systems for a long time. There are used for constructing the linear predicates on the set of masks for perceptrons and for synthesis of threshold elements in [1,2]. In [3,4] the results of pseudo-Boolean and integer programming were considered. An attempt to use the digital signal processing methods to this branch was made in [5]. The so-called spectral methods were developed in [6-11] and were a basis for solving a number of questions of fault detection in switching circuits [12-14]. Papers [15-17] disclosed some new features of arithmetical logic which allow to solve a number of applied problems, in particular Petri networks modelling, in a new way.

The present paper is devoted to both the ways of synthesis of a family of arithmetical canonical forms with fixed polarity. This family is an analog of the Reed-Muller canonical expansions with fixed polarity [18] synthesized by spectral, or orthogonal transform methods [18,19]. The same result may be obtained using the so-called Taylor logic expansion whose coefficients are the values of some Boolean Differences [20-22]. In [10,11,23,24] the matrix algorithms to compute the Boolean Differences were created. Thus, the coefficients of a Reed-Muller expansion for a Boolean function may be calculated by means of orthogonal transforms or matrix operators of the Boolean Differentiation. The connection between matrices of the orthogonal transform and Boolean Differentiation was established as well [10].

The question about an arithmetical analog of Taylor logic expansion and Derivative for Boolean function was considered in [25], and some definitions of these derivatives were given under the names of Arithmetical Derivatives [26] and partial Gibbs derivative on finite dyadic group [27-30]. The conception of multi-polarity Arithmetical transform for Boolean function and its calculating using decision diagrams and dispoint cube were introduced in [31-33].

A synthesis of arithmetical polynomials for k-valued logic functions by the substitution method was considered in [34], and using the orthogonal transforms for this was studied in [11]. In [35-38] there were received matrix algorithms to compute a family of generalized Reed-Muller canonical (non-arithmetical) expansions for MVL functions and in [36-38] some definitions of k-valued logic derivative was given. But the question about their arithmetical analogs is actually and dictated by such applied problems as digital devices diagnostics [39,40], modelling of multistables and neural elements [41] and others.

Here we draw parallels between Reed-Muller expansion and arithmetical forms for Boolean and MVL functions. It means that the paper is constructed on the principle of analogs We consider the spectral method to synthesize these forms and introduce a definition of Arithmetical derivative for MVL-functions basing the connection between orthogonal transform matrices and matrices of these derivatives.

II. LOGIC AND ARITHMETICAL POLYNOMIAL FORMS FOR BOOLEAN FUNCTIONS

Any Boolean function $f(X) = f(x_1, x_2, \ldots, x_n)$ of n variables raises a family of 2^n canonical expansions called fixed-polarity Reed-Muller forms [10,11,18,36,37]

$$F_c(X) = \sum_{i=0}^{2^n-1} f_c^{(i)} (x_1 \oplus c_1)^{i_1} (x_2 \oplus c_2)^{i_2} \ldots (x_n \oplus c_n)^{i_n} \text{ over GF(2)} \quad (1)$$

where $c \in \overline{0, 2^n - 1}$ is the value defining the polynomial's type so that

$$(x_j \oplus c_j)^{i_j} = \begin{cases} x_j \text{ when } c_j = 0, i_j = 1; \\ \overline{x_j} \text{ when } c_j = 1, i_j = 1; \\ 1 \text{ when } c_j \in (0,1), i_j = 0; \end{cases}$$

$c_1 c_2 \ldots c_n$ is the binary representation of c; $j = \overline{1,n}$.. It was well-known discrete orthogonal transform to calculate the vector F_c of the coefficients $f_c^{(j)}$

$$F_c = K_{2^n}^{(c)} X \quad \text{over GF(2)},$$
$$X = (K_{2^n}^{(c)})^{-1} F_c \quad \text{over GF(2)}.$$

where $X = [x^{(0)} x^{(1)} \ldots x^{(2^n-1)}]$ is truth table column vector of the Boolean function $f(X)$; $K_{2^n}^{(c)}$ is the $2^n \times 2^n$ direct transform formed by the Kronecker product

$$K_{2^n}^{(c)} = K_2^{(c_1)} \otimes K_2^{(c_2)} \otimes \ldots \otimes K_2^{(c_n)},$$

the elements of the 2×2 matrix $K_2^{(c_j)}$ are found as result of solving the system of Boolean equations $B_2^{(c_j)} K_2^{(c_j)} = I_2$ over GF(2), I_2 is the 2×2 identity matrix; the elements of the matrix $B_2^{(c_j)}$ are formed as follows $b_{1,p} = (l \oplus c_j)^p = l^p$ when $c_j = 0$; $b_{1,p} = (l \oplus c_j)^p = \bar{l}^p$ when $c_j = 1$; $(K_{2^n}^{(c)})^{-1} = B_{2^n}^{(c)}$,

$$K_2^{(0)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad K_2^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The analog of expression (1) in arithmetical logic is the following canonical form [10,11]

$$P_c(X) = \sum_{i=0}^{2^n-1} p_c^{(i)} (x_1 \oplus c_1)^{i_1} (x_2 \oplus c_2)^{i_2} \ldots (x_n \oplus c_n)^{i_n} \quad (2)$$

where $p_c^{(j)}$ are the coefficients of the polynomial $P_c(X)$ formed the coefficients column vector $P_c = [p_c^{(0)} p_c^{(1)} \ldots p_c^{(2^n-1)}]$. This vector is computed by means of the orthogonal transform

$$P_c = \widetilde{K}_{2^n}^{(c)} X \ , \qquad (3)$$
$$X = (\widetilde{K}_{2^n}^{(c)})^{-1} P_c \ .$$

where matrix $\widetilde{K}_{2^n}^{(c)}$ is formed by the rule similar to $K_{2^n}^{(c)}$, $(\widetilde{K}_{2^n}^{(c)})^{-1} = B_{2^n}^{(c)}$,

$$\widetilde{K}_2^{(0)} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad \widetilde{K}_2^{(1)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

For example, for the Boolean function $f(X) = x_1 x_2 \vee x_3$, n=3, whose truthtable column vector is $X = [01010111]$ the family of the arithmetical polynomial $P_c(X)$ is represented by 2^3=8 ones

$$P_c(X) = \sum_{i=0}^{7} p_c^{(i)} (x_1 \oplus c_1)^{i_1} (x_2 \oplus c_2)^{i_2} (x_3 \oplus c_3)^{i_3} ..$$

The coefficients column vector P_c is computed using formula (3). In particular when c=3 we obtain

$$P_3 = \widetilde{K}_{2^3}^{(3)} X = [\,1\ {-1}\ 0\ 0\ 0\ 1\ 0\ {-1}]$$

and the symbolic formula is written as

$$P_3(X) = 1 - \bar{x}_3 + x_1 \bar{x}_3 - x_1 \bar{x}_2 \bar{x}_3 \,.$$

III. LOGIC AND ARITHMETICAL DERIVATIVES FOR BOOLEAN FUNCTION

The different interpretation of the coefficients $f_c^{(i)}$ in (1) results in various names of the polynomials $F_c(X)$. Thus, if these coefficients are values of Boolean Differences, then the Reed-Muller canonical expansion (1) is referred to as logic Taylor series [20-22]. Hence, the method of Boolean Differences is an alternative method to form the logic polynomials. The coefficients $f_c^{(i)}$ in expression (1) are the values of the q-order Boolean Differences of the function f(X) in the point $C \in \overline{0, 2^n - 1}$

$$f_c^{(i)} = \left. \frac{\partial^{(q)} f(X)}{\partial x_1^{i_1}\, \partial x_2^{i_2} \ldots \partial x_n^{i_n}} \right|_{X=C}, \quad \partial x_j^{i_j} = \begin{cases} 1 & \text{when } i_j = 0, \\ \partial x_j & \text{when } i_j = 1. \end{cases}$$

where q is the number of "1" in the binary representation $i_1, i_2, \ldots, i_n$ of $i, i = \overline{0, 2^n - 1}$.

Recall that the Boolean Difference of a Boolean function $f(X)$ with respect to a variable x_k is defined by [22]

$$\partial f(X)/\partial x_k = f(x_1, \ldots, x_k, \ldots, x_n) \oplus f(x_1, \ldots, \bar{x}_k, \ldots, x_n).$$

As alternative to the well-known results [10,11,23-24] to compute the Boolean Differences the matrix methods were worked out. Thus, a truth table column vector of the Boolean Difference with respect to a variable x_k for a function $f(X)$ given by its truthtable column vector X is computed by the formula

$$\partial X / \partial x_k = D_{2^n}^{(k)} X \qquad \text{over GF(2)}$$

where $D_{2^n}^{(k)}$ is the $2^n \times 2^n$ matrix formed by the rule

$$D_{2^n}^{(k)} = I_{2^{k-1}} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes I_{2^{n-k}} \,.$$

Using the results of computing the q-order Boolean Differences in the vector form the coefficients vectors F_c of the polynomials $F_c(X)$ can be formed, and vice versa.

Consider an analog of Boolean Difference in arithmetical logic. It may be obtained when interpreting polynomial (1) as arithmetical-logic Taylor series where

$$p_c^{(i)} = \left. \frac{\widetilde{\partial}^{(q)} f(X)}{\widetilde{\partial} x_1^{i_1}\, \widetilde{\partial} x_2^{i_2} \ldots \widetilde{\partial} x_n^{i_n}} \right|_{X=C}, \qquad (4)$$

Then an arithmetical derivative of a Boolean function $f(X) = f(x_1, x_2, \ldots, x_n)$ with respect to a variable x_k is of the form [26]

$$\widetilde{\partial} f(X) / \widetilde{\partial} x_k = -f(x_1, .., x_k, .., x_n) + f(x_1, .., \bar{x}_k, .., x_n). \ (5)$$

It follows from (5) that the arithmetical derivative may have the values 0, 1,-1 and indicates the direction of change of the function, i.e. it is equal 0 if $f(X)$ is not changed when changing the value of variable from x_k to $\bar{x}_k$, it is equal 1 if value of $f(X)$ increases from 0 to 1, and it is equal -1 if value of $f(X)$ decreases from 1 to 0.

For example, compute the arithmetical derivative with respect to variable x_1 for the Boolean function $f(X)=\bar{x}_1x_2\vee x_3$. Using the equation $x_1\vee x_2=x_1+x_2-x_1x_2$ we receive

$$\tilde{\partial}f(X)/\tilde{\partial}x_1=-(\bar{x}_1x_2\vee x_3)+(x_1x_2\vee x_3)=-\bar{x}_1x_2\bar{x}_3+x_1x_2\bar{x}_3.$$

The matrix analog of expression (5) is defined by the formula [26]

$$\tilde{\partial}X/\tilde{\partial}x_k=A^{(k)}_{2^n}X, \qquad (6)$$

where $2^n\times 2^n$ matrix $A^{(k)}_{2^n}$ is formed by the rule

$$A^{(k)}_{2^n}=I_{2^{k-1}}\otimes\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}\otimes I_{2^{n-k}}.$$

The result of computing the arithmetical derivative $\tilde{\partial}f(X)/\tilde{\partial}x_k$ in symbolic form is defined by an algebraic sum of some minterms m_i:

$$f^*(x_1,x_2,\dots,x_n)=\sum_i m_i,$$

which is an analogue of notation "sum of products" for a Boolean function. The minterms in this notation are simply found from the elements of truth table column vector $\tilde{\partial}X/\tilde{\partial}x_k$ [26].

For example, let us compute the arithmetical derivative with respect to variable x_1 of the Boolean function $f(X)=\bar{x}_1x_2\vee x_3$ given by its truth table column vector X=[0 1 1 1 0 1 0 1]. We use expression (6):

$$\frac{\tilde{\partial}X}{\tilde{\partial}x_1}=A^{(1)}_{2^3}X=\begin{bmatrix}-1&&&&1&&&\\&-1&&&&1&&\\&&-1&&&&1&\\&&&-1&&&&1\\1&&&&-1&&&\\&1&&&&-1&&\\&&1&&&&-1&\\&&&1&&&&-1\end{bmatrix}\begin{bmatrix}0\\1\\1\\1\\0\\1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\-1\\0\\0\\0\\1\\0\end{bmatrix}.$$

The values (-1) and 1 in the result vector correspond to the minterms $\bar{x}_1x_2\bar{x}_3$ and $x_1x_2\bar{x}_3$. Then $\tilde{\partial}f(X)/\tilde{\partial}x_1=-\bar{x}_1x_2\bar{x}_3+x_1x_2\bar{x}_3$.

A m-ordered arithmetical derivative with respect to variables $x_{i_1},x_{i_2},\dots,x_{i_m}\subset(x_1,x_2,\dots,x_n)$ of a Boolean function $f(X)=f(x_1,x_2,\dots,x_n)$ is defined in matrix form as

$$\frac{\tilde{\partial}^{(m)}X}{\tilde{\partial}x_{i_1}\tilde{\partial}x_{i_2}\dots\tilde{\partial}x_{i_m}}=A^{(i_m)}_{2^n}\dots A^{(i_2)}_{2^n}A^{(i_1)}_{2^n}X.$$

For example, let us synthesize arithmetical polynomials for the Boolean function $f(X)=x_1x_2\vee x_3$ by the method of arithmetical-logic Taylor expansion. In accordance with expressions (2) and (5) when n=3 the general description of the family of the polynomials is:

$$P_c(X)=\sum_{i=0}^{7}p_c^{(i)}(x_1\oplus c_1)^{i_1}(x_2\oplus c_2)^{i_2}(x_3\oplus c_3)^{i_3}..$$

and, in particular, when c=3 we obt:ain

$$P_3(X)=\sum_{i=0}^{7}f_3^{(i)}(x_1\oplus 0)^{i_1}(x_2\oplus 1)^{i_2}(x_3\oplus 1)^{i_3}=$$

$$=f(3)+\left.\frac{\tilde{\partial}f(x)}{\tilde{\partial}x_3}\right|_{X=3}\bar{x}_3+\left.\frac{\tilde{\partial}f(x)}{\tilde{\partial}x_2}\right|_{X=3}\bar{x}_2+$$

$$+\left.\frac{\tilde{\partial}^{(2)}f(x)}{\tilde{\partial}x_2\tilde{\partial}x_3}\right|_{X=3}\bar{x}_2\bar{x}_3+\left.\frac{\tilde{\partial}f(x)}{\tilde{\partial}x_1}\right|_{X=3}x_1+\left.\frac{\tilde{\partial}^{(2)}f(x)}{\tilde{\partial}x_1\tilde{\partial}x_3}\right|_{X=3}x_1\bar{x}_3+$$

$$+\left.\frac{\tilde{\partial}^{(2)}f(x)}{\tilde{\partial}x_1\tilde{\partial}x_2}\right|_{X=3}x_1\bar{x}_2+\left.\frac{\tilde{\partial}^{(3)}f(x)}{\tilde{\partial}x_1\tilde{\partial}x_2\tilde{\partial}x_3}\right|_{X=3}x_1\bar{x}_2\bar{x}_3.$$

The results of the matrix-vector computations of the truthtable column vectors of derivatives and arithmetical polynomials are shown in Table 1.

IV. ARITHMETICAL ANALOG OF GENERALIZED REED-MULLER EXPANSION FOR K-VALUED FUNCTIONS

A k-valued function $f(X)$ of n variables when k is a prime raises a family of k^n Reed-Muller canonical forms [37,38]

$$F_c(X)=\sum_{j=0}^{k^n-1}f_c^{(j)}(x_1+c_1)^{i_1}(x_2+c_2)^{i_2}\dots(x_n+c_n)^{i_n}(\mathrm{mod}\,k)$$

$$(x_j+c_j)^{i_j}=\begin{cases}(x_j+c_j)^{i_j}(\mathrm{mod}\ \mathrm{k}), i_j\in(\overline{1,k-1}),\\ 1\ ,\ c_j\in(0,1), i_j=0.\end{cases}$$

Here $c_1c_2\dots c_n$ and $j_1j_2\dots j_n$ are correspondingly k-valued representations of c and j; $x_j+c_j=\overset{c_j}{\hat{x}_j}$ denotes c_j-order cyclic inversion of x_j; $j=\overline{1,n}$; $f_c^{(i)}$ are the polynomial's coefficients.

An analog of the Reed-Muller canonical expansion of MVL function in arithmetical logic is written as

$$P_c(X)=\frac{1}{(k-1)^n}\sum_{j=0}^{k^n-1}p_c^{(j)}(x_1+c_1)^{i_1}(x_2+c_2)^{i_2}..(x_n+c_n)^{i_n}. \qquad (7)$$

Expression (6) is an arithmetical-logic series of a MVL-function $f(X)$ in the point c, or *arithmetical polynomial*. The computation of the coefficients $p_c^{(j)}$ may be executed by the method of orthogonal transforms by the anlogy of logic derivative.

The pair of orthogonal transform to compute spectral coefficients $p_c^{(i)}$ of the arithmetical polynomial $P(X)$ and to solve the inverse problem of recognizing the values of an assumed function $f(X)$ of n variables given by its truthtable column vector X is written as

$$P_c = \frac{1}{(k-1)^n} K_{k^n}^{(c)} X, \qquad (8)$$
$$X = B_{k^n}^{(c)} P_c.$$

Here $P_c = [p_c^{(0)} p_c^{(1)} \dots p_c^{(k^n-1)}]$ is the spectral coefficient vector of the polynomial $P_c(X)$; $K_{k^n}^{(c)}$ and $B_{k^n}^{(c)}$ is the $k^n \times k^n$ direct and inverse transform matrices formed by the rule

$$K_{k_n}^{(c)} = K_k{}^{(c_1)} \otimes K_k{}^{(c_2)} \otimes \dots \otimes K_k{}^{(c_n)}, \qquad (9)$$

where $K_k^{(c_j)}$ is the $k \times k$ elementary matrix computed as result of solving the logic equation system

$$B_k^{(c_j)} K_k^{(c_j)} = I_k, \qquad (10)$$

The element $b_{l,p}$ of the matrix $B_k^{(c_j)}$ is formed by the rule

$b_{l,p} = (l+c_i)^p, l, p \in \overline{0, k-1};$

The matrices $B_k^{(c_j)}$ and $K_k^{(c_j)}$ when k=3, $c_j = 0,1,2$, received as result of solving logic equation (10) are presented in Table 2.

Table 2.
Elementary orthogonal transforms matrices, k = 3

c_j	$B_3^{(c_j)}$	$K_3^{(c_j)}$
0	$B_3^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$	$K_3^{(0)} = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix}$
1	$B_3^{(1)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$K_3^{(1)} = \begin{bmatrix} 0 & 0 & 2 \\ 4 & -1 & -3 \\ -2 & 1 & 1 \end{bmatrix}$
2	$B_3^{(2)} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$K_3^{(2)} = \begin{bmatrix} 0 & 2 & 0 \\ -1 & -3 & 4 \\ 1 & 1 & -2 \end{bmatrix}$

Let us for the 3-valued function $f(X)$ of two variables whose truthtable vector is $X = [\ 0\ 1\ 0\ 2\ 1\ 1\ 2\ 0\ 2\ ]$ find the family of the polynomial logic forms. In accordance with (6) write for n=2:

$$P_c(X) = (1/4) \sum_{i=0}^{8} p_c{}^{(i)} (x_1 + c_1)^{i_1} (x_2 + c_2)^{i_2}$$

Execute computations when c=2 $(c_1 c_2 = 02)$. For this compute the coefficient column vector P_2 in accordance with (8) where the matrix $K_{3^2}^{(2)}$ is found using equation (9):

$$P_2 = \frac{1}{4} K_{3^2}^{(2)} X = \frac{1}{4} \left(K_3^{(0)} \otimes K_3^{(2)} \right) X =$$

$$= \frac{1}{4} \left(\begin{bmatrix} 2 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 2 & 0 \\ -1 & -3 & 4 \\ 1 & 1 & -2 \end{bmatrix} \right) X =$$

$$= \frac{1}{4} \left[\begin{array}{ccc|ccc|ccc} 0 & 4 & 0 & & & & & & \\ -2 & -6 & 8 & & & & & & \\ 2 & 2 & -4 & & & & & & \\ \hline 0 & -6 & 0 & 0 & 8 & 0 & 0 & -2 & 0 \\ 3 & 9 & -12 & -4 & -12 & 16 & 1 & 3 & -4 \\ -3 & -3 & 6 & 4 & 4 & -8 & -1 & -1 & 2 \\ \hline 0 & 2 & 0 & 0 & -4 & 0 & 0 & 2 & 0 \\ -1 & -3 & 4 & 2 & 6 & -8 & -1 & -3 & 4 \\ 1 & 1 & -2 & -2 & -2 & 4 & 1 & 1 & -2 \end{array} \right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 1 \\ 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 \\ -6 \\ 2 \\ 2 \\ -1 \\ 3 \\ -2 \\ 5 \\ -3 \end{bmatrix}$$

And finally receive

$$P_2(X) = \frac{1}{4} \sum_{i=0}^{8} p_c^{(i)} (x_1 + 0)^{i_1} (x_2 + 2)^{i_2} =$$
$$= \frac{1}{4} (4 - 6\hat{\hat{x}}_2 + 2\hat{\hat{x}}_2^2 + 2x_1 - x_1\hat{\hat{x}}_2 + 3x_1\hat{\hat{x}}_2{}^2 -$$
$$-2\hat{\hat{x}}_1{}^2 + 5x_1^2\hat{\hat{x}}_2 - 3x_1{}^2\hat{\hat{x}}_2{}^2).$$

It should be noted a one significant circumstance for realizing the algorithm. The matrix $K_{k^n}^{(c)}$ may be factorized, i.e. represented as product of n sparse matrices

$$K_{k^n}^{(c)} = T_{k^n}^{(1)} T_{k^n}^{(2)} \dots T_{k^n}^{(n)},$$

where

$$T_{k^n}^{(i)} = I_{k^{n-i}} \otimes K_k^{(c_i)} \otimes I_{k^{i-1}}, i = \overline{1,n}.$$

It means that the algorithm to compute vector P_c is fast discrete orthogonal transform of n -iterations.

V. ARITHMETICAL DERIVATIVES OF MVL FUNCTIONS

Interpret formula (7) as arithmetical analogue of generalized Taylor logic expansion for MVL-functions. Then the coefficients $p_c^{(j)}$ of the polynomial $P_c(X)$ are to be arithmetical analogues of the coefficients $f_c^{(j)}$ of logic Taylor series (Reed-Muller canonical expansion) $F(X)$. In $F(X)$ the coefficients $f_c^{(j)}$ are the values of q-ordered logic Derivative in the point (k-c), i.e. when $x_1 = k - c_1$, $x_2 = k - c_2$,..., $x_n = k - c_n$ [37,38]

$$f_c{}^{(i)} = \left. \frac{\partial^{(q)} f(X)}{\partial \hat{x}_1{}^{k-i_1} \partial \hat{x}_2{}^{k-i_2} \dots \partial \hat{x}_n{}^{k-i_n}} \right| X = (k-c)$$

$$\partial \hat{x}_j^{k-i_j} = \begin{cases} 1 & , k = i_j, \\ \partial \hat{x}_j^{k-i_j} & , k \neq i_j. \end{cases}$$

Here q is the number of non-zero digits of k-valued representation of $(k-c_1, k-c_2, \dots, k-c_n)$.

Write the i-th coefficient $p_c^{(j)}$ of the polynomial $P_c(X)$ (7) as

$$p_c^{(i)} = \left.\frac{\tilde{\partial}^{(q)} f(X)}{\tilde{\partial}x_1^{k-i_1}\tilde{\partial}x_2^{k-i_2}\ldots\tilde{\partial}x_n^{k-i_n}}\right| X = (k-c)$$

$$\tilde{\partial}\hat{x}_j^{k-i_j} = \begin{cases} 1 & ,k = i_j, \\ \tilde{\partial}\hat{x}_j^{k-i_j} & ,k \neq i_j. \end{cases}$$

We will call it the *q-ordered arithmetical Derivative of a MVL-function f(X).*

Proceeding from the connection of orthogonal transform matrices $\tilde{K}_{2^n}^{(c)}$ and matrices $A_{2^n}^{(i)}$ for Boolean functions we have received the matrix interpretation of q-ordered arithmetical Derivative for k-valued function $f(X)$ given by its truthtable column vector X:

$$\frac{\tilde{\partial}^{(q)} X}{\tilde{\partial}\hat{x}_1^{r_1}\tilde{\partial}\hat{x}_2^{r_2}\ldots\tilde{\partial}\hat{x}_n^{r_n}} = \frac{1}{(k-1)^n}\overset{r_n}{\hat{A}}{}_{k^n}^{(n)}\ldots\overset{r_2}{\hat{A}}{}_{k^n}^{(2)}\,\overset{r_1}{\hat{A}}{}_{k^n}^{(1)}X, \quad (11)$$

where matrix $\overset{r_j}{\hat{A}}{}_{k^n}^{(j)}$ is formed by the rule

$$\overset{r_j}{\hat{A}}{}_{k^n}^{(j)} = (k-1)I_{k^{j-1}} \otimes \left(\sum_{p=0}^{k-1} K_{k-r_j,p} I_k^{(p\rightarrow)}\right) \otimes I_{k^{n-j}}. \quad (12)$$

Consequently, the matrices $\overset{r_j}{\hat{A}}{}_{k^n}^{(j)}$ may be formed from the matrices $K_{k^n}^{(c)}$. Fig.1 illustrates that for k=3, n=1.

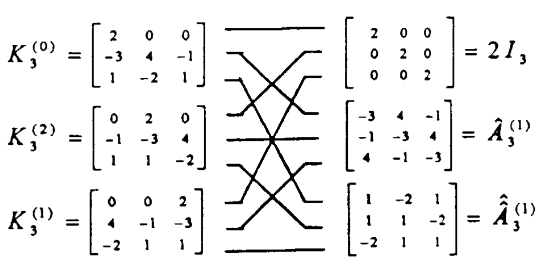

Fig.1. The connection between matrices $K_3^{(c)}$ and $\overset{r_1}{\hat{A}}{}_3^{(1)}, r_1 = 1,2.$

In (12) $I_k^{(p\rightarrow)}$ is the $k \times k$ matrix obtained by means of cyclic right shift of the elements of identity matrix I_k; $K_{k-r_j,p}$ is the $(\mathrm{k}-\mathrm{r_j}, p)$-th element of the matrix $K_k^{(0)}$.

It should be noted that on the view of digital signal processing the logic and Arithmetical derivatives of a MVL function may be considered as correlations between truth table vectors of the function and the basis functions (the columns of the orthogonal transform matrice). This accounts for the connection between these and differentiation matrices.

When in (10) suppose q=1 we receive the matrix interpretation of *an Arithmetical derivative of a k-valued function $f(X)$ with respect to a variable x_j with r_j-order cyclic inversion*

$$\partial X \Big/ \partial \overset{r_j}{\hat{x}}_j = \frac{1}{(k-1)}\overset{r_j}{\hat{A}}{}_{k^n}^{(j)} X. \quad (13)$$

When k=2 expression (13) coincides with definition of the matrix form of arithmetical Derivative for a Boolean function (6).

Proceeding from matrix definition of the arithmetical Derivative (13) we introduce the following definition of the arithmetical Derivative with respect to a variable x_j with r_j-ordered cyclic inversion in symbolic form:

$$\tilde{\partial} f(X) \Big/ \tilde{\partial}\overset{r_j}{\hat{x}}_j = \sum_{p=0}^{k-1} K_{k-r_j,p} f(x_1,\ldots,\overset{p}{\hat{x}}_j,\ldots,x_n). \quad (14)$$

When k=2 expression (14) comes to the definition of the arithmetical Derivative with respect to a variable for a Boolean function (5).

For example, write various arithmetical Derivatives of a function $f(X)$ of n variables when k=3. Using formula (14) for k=3 we receive

$$\tilde{\partial} f(X) \Big/ \tilde{\partial}\overset{r_j}{\hat{x}}_j = \sum_{p=0}^{2} K_{3-r_j,p} f(x_1,..,\overset{p}{\hat{x}}_j,..,x_n).$$

Since $r_j = \overline{1,k-1} = \overline{1,2}$, then there are two arithmetical Derivatives with respect to a variable x_j of a 3-valued function $f(X)$:

- with 1-ordered cyclic inversion

$$\tilde{\partial} f(X) \Big/ \tilde{\partial}\hat{x}_j = f(x_1,\ldots,x_j,\ldots,x_n) -$$
$$-2f(x_1,\ldots,\hat{x}_j,\ldots,x_n) + f(x_1,\ldots,\hat{\hat{x}}_j,\ldots,x_n),$$

-with 2-ordered cyclic inversion

$$\tilde{\partial} f(X) \Big/ \tilde{\partial}\hat{\hat{x}}_j = -3f(x_1,\ldots,x_j,\ldots,x_n) +$$
$$+4f(x_1,\ldots,\hat{x}_j,\ldots,x_n) - f(x_1,\ldots,\hat{\hat{x}}_j,\ldots,x_n).$$

For example, compute the arithmetical Derivative $\partial f(X) \Big/ \partial\hat{\hat{x}}_2$ for the function $f(X) = x_1 + (x_1 \wedge x_2)$ (mod 3) given by the truthtable column vector $X = [000122201]$.

In accordance with (12), (13) write

$$\frac{\tilde{\partial} X}{\tilde{\partial}\hat{\hat{x}}_2} = \frac{1}{4}\hat{A}_{3^2}^{(2)} X =$$

$$= \frac{1}{4}(I_3 - 2I_3^{(1\rightarrow)} + I_3^{(2\rightarrow)}))X =$$

$$=\frac{1}{4}\left(\begin{bmatrix}2 & & \\ & 2 & \\ & & 2\end{bmatrix}\otimes\begin{bmatrix}1 & -2 & 1\\ 1 & 1 & -2\\ -2 & 1 & 1\end{bmatrix}\right)X=$$

$$=\left[\begin{array}{ccc|ccc|ccc}2 & -4 & 2 & & & & & & \\ 2 & 2 & -4 & & & & & & \\ -4 & 2 & 2 & & & & & & \\ \hline & & & 2 & -4 & 2 & & & \\ & & & 2 & 2 & -4 & & & \\ & & & -4 & 2 & 2 & & & \\ \hline & & & & & & 2 & -4 & 2\\ & & & & & & 2 & 2 & -4\\ & & & & & & -4 & 2 & 2\end{array}\right]\begin{bmatrix}0\\1\\0\\2\\1\\1\\2\\0\\2\end{bmatrix}=\begin{bmatrix}-4\\2\\2\\2\\2\\-4\\8\\-4\\-4\end{bmatrix}.$$

Now let us demonstrate the technique to synthesize arithmetical canonical form of a MVL-function by using the Arithmetical derivative.

For example, synthesize the family of arithmetical polynomials of a 3-valued function given by its truthtable column vector $X = [\ 0\ 1\ 0\ 2\ 1\ 1\ 2\ 0\ 2\ ]$. This family includes 9 arithmetical polynomial (7) whose coefficients are computed by means of the arithmetical Derivatives

$$p_c^{(j)} = \tilde{\partial}^{(q)} f(3-c_1, 3-c_2) \,/\, \tilde{\partial}^{3-j_1}\hat{x}_1\, \tilde{\partial}^{3-j_2}\hat{x}_2 .$$

The matrix form of this Derivative is founded from (11)

$$\frac{\tilde{\partial}^{(2)}X}{\tilde{\partial}x_1^{r_1}\,\tilde{\partial}x_2^{r_2}} = \frac{1}{4}\,\overset{r_2}{\hat{A}}{}_9^{(2)}\ \overset{r_1}{\hat{A}}{}_9^{(1)}\,X,$$

The results of computations are presented by Table 3. To find the symbolic notation of the arithmetical canonical forms it is needed to substitute the coefficients received into (6):

$$P_0(X) = 1/4(8x_2 - 4x_2^2 + 12x_1 -$$
$$- 16x_1x_2 + 6x_1x_2^2 - 4x_1^2 + 2x_1^2x_2);$$
$$P_1(X) = 1/4(-2\hat{\hat{x}}_2 + 2\hat{\hat{x}}_2^2 + 4x_1 +$$
$$+ 17x_1\hat{\hat{x}}_2 - 9x_1\hat{\hat{x}}_2^2 - 7x_1^2\hat{\hat{x}}_2 + 3x_1^2\hat{\hat{x}}_2^2);$$
$$P_2(X) = 1/4(4 - 6\hat{x}_2 + 2\hat{x}_2^2 - x_1\hat{x}_2 +$$
$$+ 3x_1\hat{x}_2^2 - 2x_1^2 + 5x_1^2\hat{x}_2 - 3x_1^2\hat{x}_2^2);$$
$$P_3(X) = 1/4(8 - 16x_2 + 8x_2^2 - 16\hat{x}_1 + 43\hat{x}_1x_2 -$$
$$-21\hat{x}_1x_2^2 + 8\hat{x}_1^2 - 19\hat{x}_1^2x_2 + 9\hat{x}_1^2x_2^2);$$
$$P_4(X) = 1/4(8 + 4\hat{x}_2 - 4\hat{x}_2^2 - 14\hat{x}_1 - 14\hat{x}_1\hat{x}_2 +$$
$$+12\hat{x}_1\hat{x}_2^2 + 6\hat{x}_1^2 + 8\hat{x}_1^2\hat{x}_2 - 6\hat{x}_1^2\hat{x}_2^2);$$
$$P_5(X) = 1/4(12\hat{\hat{x}}_2 - 4\hat{\hat{x}}_2^2 + 6\hat{x}_1 -$$
$$-29\hat{x}_1\hat{\hat{x}}_2 + 9\hat{x}_1\hat{\hat{x}}_2^2 - 2\hat{x}_1^2 + 11\hat{x}_1^2\hat{\hat{x}}_2 + 3\hat{x}_1^2\hat{\hat{x}}_2^2);$$
$$P_6(X) = 1/4(8 - 6x_2 + 2x_2^2 + 4\hat{\hat{x}}_1 -$$
$$-27\hat{\hat{x}}_1x_2 + 15\hat{\hat{x}}_1x_2^2 - 4\hat{\hat{x}}_1^2 + 17\hat{\hat{x}}_1^2x_2 - 9\hat{\hat{x}}_1^2x_2^2);$$
$$P_7(X) = 1/4(4 + 8\hat{x}_2 - 4\hat{x}_2^2 + 10\hat{\hat{x}}_1 -$$
$$-3\hat{\hat{x}}_1\hat{x}_2 - 3\hat{\hat{x}}_1\hat{x}_2^2 - \hat{\hat{x}}_1^2 - \hat{\hat{x}}_1^2\hat{x}_2 + 2\hat{\hat{x}}_1^2\hat{x}_2^2);$$
$$P_8(X) = 1/4(4 - 2\hat{\hat{x}}_2 + 2\hat{\hat{x}}_2^2 - 8\hat{\hat{x}}_1 +$$
$$+30\hat{\hat{x}}_1\hat{\hat{x}}_2 - 12\hat{\hat{x}}_1\hat{\hat{x}}_2^2 + 4\hat{\hat{x}}_1^2 - 16\hat{\hat{x}}_1^2\hat{\hat{x}}_2 + 6\hat{\hat{x}}_1^2\hat{\hat{x}}_2^2).$$

Thus, the coefficients of polynomials $P_c(X)$ may be computed by two methods, i.e. orthogonal transform and Arithmetical derivatives of MVL functions.

VI. SUMMARY AND CONCLUSIONS

In this paper a problem of synthesis of arithmetical canonical forms for MVL-functions proposed to solve by two methods. Both the methods are formalized as matrix transforms. The main reason to choose the matrix form is that the algorithms synthesized on this base are mapped into parallel architectures. The principles and methods to design such architectures, in particular, the linear systolic arrays, to realize the fast orthogonal transforms and matrix form of logic derivatives were considered in [10,11,24,37] were proposed. Furthermore the connection between the considered orthogonal transform and differentiation procedures that is established through the arithmetical-logic Taylor expansion, allows to compute the polynomials using hardware for differentiation and vice versa.

However, the algorithms received are realisable by new architectures and program packages too. Thus, in [41] there were considered neural-based algorithms of computing MVL functions presented by the arithmetical polynomials. Thus these polynomial forms are the logic functions models which meet the requirements of realisable on neural nets.

Furthermore, the received new definition of Arithmetical derivative is of independent importance too. Generalizing the results received in [14,39,40] it may be considered as tools for fault detection in k-valued circuits.

Thus the arithmetical analogs of the Reed-Muller expansion are means to decide a class of applied problems supported by methods of algebra of MVL functions.

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Table 1.

Values of arithmetical derivatives and their connection with values of arithmetical polinomial coefficients for the Boolean function $f(X) = x_1 x_2 \vee x_3$

		i	0	1	2	3	4	5	6	7
c	$c_1c_2c_3$	$i_1i_2i_3$	000	001	010	011	100	101	110	111
		P_c	X	$\frac{\tilde{\partial} X}{\tilde{\partial} x_3}$	$\frac{\tilde{\partial} X}{\tilde{\partial} x_2}$	$\frac{\tilde{\partial}^{(2)} X}{\tilde{\partial} x_2 \tilde{\partial} x_3}$	$\frac{\tilde{\partial} X}{\tilde{\partial} x_1}$	$\frac{\tilde{\partial}^{(2)} X}{\tilde{\partial} x_1 \tilde{\partial} x_3}$	$\frac{\tilde{\partial}^{(2)} X}{\tilde{\partial} x_1 \tilde{\partial} x_2}$	$\frac{\tilde{\partial}^{(3)} X}{\tilde{\partial} x_1 \tilde{\partial} x_2 \tilde{\partial} x_3}$
0	0 0 0	P_0	0	1	0	0	0	0	1	-1
1	0 0 1	P_1	1	-1	0	0	0	0	0	1
2	0 1 0	P_2	0	1	0	0	1	-1	-1	1
3	0 1 1	P_3	1	-1	0	0	0	1	0	-1
4	1 0 0	P_4	0	1	1	-1	0	0	-1	1
5	1 0 1	P_5	1	-1	0	1	0	0	0	-1
6	1 1 0	P_6	1	0	-1	1	-1	1	1	-1
7	1 1 1	P_7	1	0	0	-1	0	-1	0	1

Table 3.

Values of arithmetical Derivatives and their connection with values of arithmetical polynomial coefficients for 3-valued function $f(X)$ given by its truthtable column vector X=[010211202]

c	c_1c_2	3-$c_1$3-c_2	j		1	2	3	4	5	6	7	8
			j_1 j_2	0	01	02	10	11	12	20	21	22
			3-$j_1$3-j_2	0	02	01	20	22	21	10	12	11
			$4P_c$	$4X$	$\frac{\tilde{\partial} X}{\tilde{\partial} \hat{x}_2}$	$\frac{\tilde{\partial} X}{\tilde{\partial} \hat{x}_2}$	$\frac{\tilde{\partial} X}{\tilde{\partial} \hat{x}_1}$	$\frac{\tilde{\partial}^{(2)} X}{\tilde{\partial} \hat{x}_1 \tilde{\partial} \hat{x}_2}$	$\frac{\tilde{\partial}^{(2)} X}{\tilde{\partial} \hat{x}_1 \tilde{\partial} \hat{x}_2}$	$\frac{\tilde{\partial} X}{\tilde{\partial} \hat{x}_1}$	$\frac{\tilde{\partial}^{(2)} X}{\tilde{\partial} \hat{x}_1 \tilde{\partial} \hat{x}_2}$	$\frac{\tilde{\partial}^{(2)} X}{\tilde{\partial} \hat{x}_1 \tilde{\partial} \hat{x}_2}$
0	00	00=0	$4P_0$	0	8	-4	12	-16	6	-4	2	0
2	02	01=1	$4P_2$	4	-6	2	2	-1	3	-2	5	-3
1	01	02=2	$4P_1$	0	-2	2	4	17	-9	0	-7	3
6	20	10=3	$4P_6$	8	-6	2	4	-27	15	-4	17	-9
8	22	11=4	$4P_8$	4	-2	2	-8	30	-12	4	-16	6
7	21	12=5	$4P_7$	4	8	-4	10	-3	-3	6	-1	3
3	10	20=6	$4P_3$	8	-16	8	-16	43	-21	8	-19	9
5	12	21=7	$4P_5$	0	12	-4	6	-29	9	-2	11	-3
4	11	22=8	$4P_4$	8	4	-4	-14	-14	12	6	8	-6

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