Proceedings of the 30th IEEE International Symposium on Multiple Valued Logic, Portland, Oregon, USA, 2000, pp.141-146

Experiments on FPRM Expressions for Partially Symmetric Logic Functions

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Abstract

This paper focuses on the fixed polarity Reed-Muller (FPRM) expression of multiple-valued logic (MVL) symmetric functions. In the FPRM expression, each variable occurs in exactly one complemented form. We show properties of the FPRM of partially symmetric functions and report experimental results for certain benchmark functions.

Keywords: MVL functions, symmetric functions, fixed polarity Reed-Muller expression.

1 Introduction

Although symmetric functions are not as well studied in MVL as in binary logic, there have been some noteworthy advances. In the first ISMVL, an identification algorithm for MVL functions was shown [4]. In the seventh Symposium, a special case of symmetric functions, called fundamental symmetric functions, was introduced [2]. MVL functions that are symmetric in both variable labels and values have been characterized in [1].

In this paper, we consider the fixed polarity Reed-Muller expression of a symmetric function. Such an expression involves the modulo sum of product terms which consists of the modulo product of variables, so that each variable appears complemented in exactly one way. For a given function, there is exactly one expression for a given polarity. The minimization problem is then one of determining which polarity yields the fewest product terms. For Boolean functions, modulo sum is exclusive OR and

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[*] *Support in part from the St. Francis Xavier University, Antigonish, Nova Scotia (CANADA) and from the State Committee of Scientific Research (POLAND) is acknowledged*

[†] *Research supported by NPS Direct Grant Funded UHCD1 (USA)*

[‡] *Research supported by the NSERC (CANADA)*

[§] *The author gratefully acknowledges the facilities provided as James Chair Professor from the St. Francis Xavier University, Antigonish, Nova Scotia (CANADA)*

modulo product is the AND function, while each variable can be complemented in exactly two ways.

2 Formulation of the problem

Given an m-valued logic function f with n variables $x_1, x_2, ..., x_n$ of which k are symmetric, we want to find the number of products (product cost) of the FPRM expression of a given polarity. Since the FPRM expression is unique, minimization is simply a process of choosing the polarity that yields the lowest cost expression.

In the paper we adopt the FPRM expression introduced by Green and Taylor [3], where variable x_i always appears in the form $x_i + p_i$, $i \in \{1, 2, ..., n\}$, $p_i \in \{0, 1, ..., m-1\}$, and $+$ is addition modulo m.

Definition 2.1 *The* **polarity vector** *of an FPRM expression F of an n-variable m-valued function f is an n-tuple of p_i's, $P = [p_1 p_2 ... p_n]$. We say that F has* **polarity** *p, where $p = p_1 m^{n-1} + p_2 m^{n-2} + ... + p_n m^0$.*

Definition 2.2 *The FPRM expression of an m-valued n-variable function f with polarity p is*

$$f_p = \sum_{j=0}^{m^n-1} r_p^{(j)} (x_1 + p_1)^{j_1} \dots (x_n + p_n)^{j_n}. \tag{2.1}$$

Here, $\sum$ and $+$ are modulo m addition, concatenation is multiplication modulo m, $r_p^{(j)}$ is a coefficient whose value is either 0, 1, $\dots$ or $m-1$, and p_i is the i-th digit in the m-ary representation $p_1 p_2 \dots p_n$ of p, i.e. $p = p_1 m^{n-1} + p_2 m^{n-2} + ... + p_n m^0$; j_i is the i-th digit in the m-ary (positional number) representation $j_1 j_2 \dots j_n$ of j. Exponentiation, as in $(x_i + p_i)^{j_i}$, is repeated multiplication. For convenience, we use $x_i + p_i = \overset{p_i}{\hat{x}}_i$, i.e. x_i, $x_i + 1 = \hat{x}_i$, $x_i + 2 = \hat{\hat{x}}_i$ and so on.

Example 2.1 *Let $m = 3$, $n = 2$, and $p = 5$. The latter implies $p_1 = 1$ and $p_2 = 2$. Thus, x_1 appears as $x_1 + p_1 = \hat{x}_1$, and x_2 appears as $x_2 + p_2 = \hat{\hat{x}}_2$, and the FPRM expression is $f = \sum_{j=0}^{8} r_5^{(j)} (x_1 + 1)^{j_1} (x_2 + 2)^{j_2}$ $= r^{(0)}\hat{x}_1^0\hat{\hat{x}}_2^0 + r^{(1)}\hat{x}_1^0\hat{\hat{x}}_2^1 + r^{(2)}\hat{x}_1^0\hat{\hat{x}}_2^2 + r^{(3)}\hat{x}_1^1\hat{\hat{x}}_2^0 + r^{(4)}\hat{x}_1^1\hat{\hat{x}}_2^1 + r^{(5)}\hat{x}_1^1\hat{\hat{x}}_2^2 + r^{(6)}\hat{x}_1^2\hat{\hat{x}}_2^0 + r^{(7)}\hat{x}_1^2\hat{\hat{x}}_2^1 + r^{(8)}\hat{x}_1^2\hat{\hat{x}}_2^2$ $= r^{(0)} + r^{(1)}\hat{\hat{x}}_2 + r^{(2)}\hat{\hat{x}}_2^2 + r^{(3)}\hat{x}_1 + r^{(4)}\hat{x}_1\hat{\hat{x}}_2 + r^{(5)}\hat{x}_1\hat{\hat{x}}_2^2 + r^{(6)}\hat{\hat{x}}_2^2 + r^{(7)}\hat{x}_1^2\hat{\hat{x}}_2 + r^{(8)}\hat{x}_1^2\hat{\hat{x}}_2^2$.*

Definition 2.3 *Given the p-polarity RM expression F of a function f, the* **polarity distribution vector** *is the m element vector $N(p) = [n_0 n_1 ... n_{m-1}]$, where n_l is the number of variables in F in the form $x + l = \overset{l}{\hat{x}}$, $l \in \{0, 1, ..., m-1\}$, i.e. n_l is the number of l's in the polarity vector P.*

The polarity distribution vector specifies some assignment of cyclic inversions to the various variables.

Definition 2.4 *For an assignment $J = j_1 j_2 ... j_n$ of values to variables, that corresponds to j-th coefficient of p-polarity RM expression, $N_0, N_1, ..., N_{m-1}$ is a set of* assignment distribution vectors, *where $N_l = [nl_0, nl_1, ..., nl_{m-1}]$, $nl_0, nl_1, ..., nl_{m-1}$ are the number of 0's, 1's, ...,$(m-1)$'s assigned to variables $x_i + l$ within J, so that to satisfy the equations $\sum_{q=0}^{m-1} nl_q = n_l$.*

Example 2.2 *Consider j-th coefficient $r_p^{(j)}$ in a FPRM expression of a 10-variable ternary function, given $P = [1022011100]$ and $J = 0202211200$. The polarity distribution vector is $N(p) = [442]$. Fig. 1 illustrates from what bits of the assignment J, the values of $nl_0, nl_1, ..., nl_{m-1}$ are formed. In that case, $N_0 = [n0_0 n0_1 n0_2] = [202]$, $N_1 = [n1_0 n1_1 n1_2] = [121]$, $N_2 = [n2_0 n2_1 n2_2] = [101]$. So, the coefficient $r_p^{(j)}$ is assigned to the following product: $(x_1 + 1)^0 x_2^2 (x_3 + 2)^0 (x_4 + 2)^2 x_5^2 (x_6 + 1)^1 (x_7 + 1)^1 (x_8 + 1)^2 x_9^0 x_{10}^0 = x_2^2 \hat{\hat{x}}_4^2 x_5^2 \hat{x}_6 \hat{x}_7 \hat{x}_8^2$.*

$$P = \underbrace{1}_{n_1}\underbrace{0}_{n_0}\underbrace{2\quad 2}_{n_2}\underbrace{0}_{n_0}\underbrace{1\quad 1\quad 1}_{n_1}\underbrace{0\quad 0}_{n_0}$$
$$J = \underbrace{0}_{n1_0}\underbrace{2}_{n0_2}\underbrace{0}_{n2_0}\underbrace{2}_{n2_2}\underbrace{2}_{n0_2}\underbrace{1\quad 1}_{n1_1}\underbrace{2}_{n1_2}\underbrace{0\quad 0}_{n0_0}$$

Figure 1. Structure of P and J

3 Partially symmetric MVL functions

An m-valued function f is *partially symmetric* with respect to k variables if and only if it is unchanged by any permutation of these variables.

Example 3.1 *Consider a 3-variable ternary function f whose truth table (column) vector is given by $[121200102$ 212001001 $120010210]^T$. $f(x_1, x_2, x_3)$ is partially symmetric in x_1 and x_3.*

Definition 3.1 *The* symmetry vector *of a function f is n-element vector $\mathbf{S} = [s_1 s_2 ... s_n]$, where $s_i = s_j = 1$ iff f is unchanged by an interchange of variables x_i and x_j.*

Example 3.2 *Consider a function that is partially symmetric in exactly one set of variables. The symmetry vector $\mathbf{S} = [101]$ specifies a function that is partially symmetric in variables $\{x_1, x_3\}$. $\mathbf{S} = [111]$ specifies a 3-variable totally symmetric function.*

4 Evaluating of FPRM expression of a partially symmetric function

To determine the product cost of an FPRM expression, we count the number of nonzero coefficients. In case of symmetric functions, a reduction in computation occurs because symmetry causes specific coefficients to be identical with other coefficients.

4.1 The number of distinct fixed polarities

The polarities of FPRM expressions can be grouped according to the number of variables whose form is x, $\widehat{x}$, $\ldots$, and $\overset{m-1}{\widehat{x}}$. Because the function is symmetric in k variables, it does not matter *which* of these variables occur in some inverted form; it matters only how many there are in each form.

Definition 4.1 A distinct group *of FPRM expressions of a function f partially symmetric in k variables $\{x_{t_1}, \ldots, x_{t_k}\}$ contains all expressions with polarity vector $P = [p_1 \ldots p_{t_1} \ldots p_{t_k} \ldots p_n]$ such that each expression includes the same number of uninverted, inverted once, twice, $\ldots$, (m-1)-times variables within $\{x_{t_1}, \ldots, x_{t_k}\}$.*

Theorem 4.1 *Given an m-valued n-variable logic function partially symmetric in k variables, the number N_d of distinct FPRM expressions is*

$$N_d = \binom{k+m-1}{m-1} m^{n-k}. \tag{4.1}$$

It is not surprising that N_d is identical to the number of functions symmetric in a single set of k variables.

Let an n-variable m-valued function be partially symmetric with respect to variables $\{x_{t_1}, ..., x_{t_k}\} \subseteq x_1, ..., x_n$. The code $p_1 p_2 ... p_n$ of polarity p for a distinct group of FPRM forms of such a function is characterized by a *symmetry distribution vector*.

Definition 4.2 *The* symmetry distribution vector *is the m element vector $K(p) = [k_0, k_1, ..., k_{m-1}]$, where $k_0 \leq n_0$, $k_1 \leq n_1$,..., $k_{m-1} \leq n_{m-1}$, and k_l is the number of l's in the corresponding k digits $p_{t_1}, p_{t_2}, ..., p_{t_k}$ of the polarity vector $P = [p_1 p_2 ... p_n]$, so that $k_0 + k_1 = ... + k_{m-1} = k$; n_l is defined from Definition 2.3.*

Example 4.1 *Given a 4-variable quaternary function symmetric with respect to variables $\{x_1, x_3\}$, i.e. the symmetry vector is $S = [1010]$, consider its FPRM expression of polarity $P = [3301]$. The symmetry distribution vector is $K(p) = [k_0, k_1, k_2, k_3] = [1, 0, 0, 1]$. If the function is totally symmetric, $K(p) = [n_0, n_1, n_2, n_3] = [1, 1, 0, 2]$.*

Note, that expressions within a distinct group of FPRM forms have to be characterized by the same symmetry distribution vector $K(p)$. Because the function is symmetric in k variables, it does not matter which of these variables are assigned with which cyclic inversions; it only matters how many inversions there are of each type. In the following result, we count the number of ways this can be done.

Theorem 4.2 *Given symmetry distribution vector $K(p) = [k_0, k_1, ..., k_{m-1}]$, the number of different p-polarity FPRM expressions within p-th distinct group, $p = 0, 1, ..., N_d$, is the multinomial of k over the partition $k_0 + k_1 + \ldots + k_{m-1} = k$*

$$N_g = \binom{k}{k_0, k_1, \ldots, k_{m-1}}. \tag{4.2}$$

Proof: The number of ways in which k_0 of k variables can be uninverted, k_1 of k can appear in the form $x + 1 = \widehat{x}$, k_2 of k can can appear in the form $x + 2 = \widehat{\widehat{x}}$, and so on, is $\binom{k}{k_0, k_1, \ldots, k_{m-1}} = k!/(k_0! k_1! ... k_{m-1}!)$.

For a totally symmetric m-valued function, $N_g = \binom{n}{n_0, n_1, \ldots, n_{m-1}}$.

Example 4.2 *For a 6-variable ternary function symmetric in 2 variables, $N_d = 486$, while the total number of polarities is $3^6 = 729$. So, each distinct polarity associates a group of N_g polarities. Here, for $p = 0 - 81, 244 - 324, 406 - 486$, $N_g = 1$, and for other polarities, $N_g = 2$. For 6-variable totally symmetric ternary function, N_g is in the range from 1 to 90.*

4.2 The number of distinct coefficients in FPRM expression

A distinct FPRM expression of a m-valued function partially symmetric in k variables is characterized by (i) *the number of distinct coefficients*, each is assigned to a group of products, (ii) *the cost of a distinct coefficient* that is the number of products in the group.

FPRM expression of an n-variable m-valued function, partially symmetric in k variables, contain several distinct groups of coefficients, and the number of the groups is counted as below.

Theorem 4.3 *Given the symmetry distribution vector $K(p) = [k_0, k_1, ..., k_{m-1}]$, the number N_c of distinct coefficients in the corresponding FPRM expression of an m-valued logic function partially symmetric with respect to k variables is*

$$N_c = m^{n-k} \prod_{l=0}^{m-1} \binom{k_l + m - 1}{m - 1} \tag{4.3}$$

Proof: *Within a product of variables* $x_1, ..., x_{t_1}, ..., x_{t_k}, ..., x_n$, k_0 *of* k *variables* $x_{t_1}, ..., x_t$ *appears uninverted (i.e. in the form* $\overset{0}{\widehat{x}} = x$*) none* $(x^0 = 1)$*, once* $(x^1 = x)$*, ...,* $(m-1)$ *times* $(x^{m-}$ *corresponds to choosing* m *objects from* k_0 *with repe by* $\binom{k_0+m-1}{m-1}$ *ways. Also,* k_1 *of these variables can a inverted (*$\overset{1}{\widehat{x}} = \widehat{x}$*) none, once ,...,* $(m-1)$ *times, that is re to* $\binom{k_1+m-1}{m-1}$ *ways. Finally, the number of ways to c* m *from* k_{m-1} *with repetition is* $\binom{k_{m-1}+m-1}{m-1}$*. Combi of these cases leads to* $\binom{k_0+m-1}{m-1}\binom{k_1+m-1}{m-1}...\binom{k_{m-1}+}{m-}$ *Taking into account that other* $(n-k)$ m*-valued vari stay fixed, we have to multiply this product by* m^{n-} *yields the equation (4.3).*

Example 4.3 *Consider a 10-variable switching fu symmetric in variables* x_2, x_3*, i.e.* $S = [011000$ *Given polarity vector* $P = [0010110011]$*, the sym distribution vector is* $K(p) = [k_0, k_1] = [1, 1]$ *. (4.3), we obtain* $N_c = 1024$*. Now, let* $S = [000111$ $K(p) = [3, 4]$*, then* $N_c = 160$ *. For totally sym function,* $N_c = (5+1)(5+1) = 36$*.*

For a totally symmetric m-valued logic function, $\prod_{l=0}^{m-1}\binom{n_l+m-1}{m-1}$.

The particular case of the code of polarity p, suc $p_1 = p_2 = ... = p_{m-1}$ corresponds to the lower bou the number of distinct coefficients of FPRM expressio partially symmetric function.

Corollary 4.1 *An* m*-valued* n*-variable function par symmetric in* k *variables can be represented with* $\binom{k+m-1}{m-1}m^{n-k}$ *distinct FPRM coefficients of pola with the polarity vector* P *such that* $p_1 = p_2 = ... =$ *For a totally symmetric function and such the pol* $N_c = \binom{n+m-1}{m-1}$ *.*

Example 4.4 *The dependence of* N_c *on the numb symmetric variables, with respect to the all possible 729 polarities, is illustrated Fig. 2. For instance, it shows us 3 minima corresponding to the beginning, last and middle polarities for ternary functions symmetric in 2 variables.*

Definition 4.3 *Let* $PS = \{x_{t_1}, x_{t_2}, ..., x_{t_k}\}$ *be the set of* k *symmetric variables in the set* $X = \{x_1, x_2, ..., x_n\}$ *and an* m*-valued function symmetric in these variables is given by its coefficient vector of* p*-polarity RM expression, and* j*-th coefficient corresponds to an assignment* $J = j_1 j_2 ... j_n$ *of values to* X*. For any assignment* J*,* $K_0(p), K_1(p), ..., K_{m-1}(p)$ *is a set of* **assignment distribution vectors**, *where* $K_l(p) = [kl_0, kl_1, ..., kl_{m-1}]$*,* $kl_0, kl_1, ..., kl_{m-1}$ *are the number of 0's, 1's, ...,*$(m-1)$*'s assigned to variables* $x_i + l$ *in* PS *within* A*, so that to satisfy the equations* $\sum_{q=0}^{m-1} kl_q = k_l, l \in (0, 1, ..., m-1)$*,* $\sum_{l=0}^{m-1} k_l = k$*.*

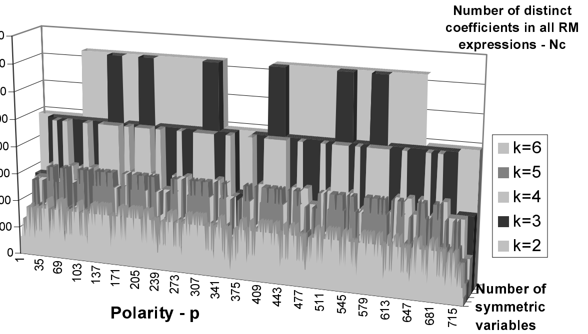

Figure 2. The number of distinct coefficients N_c in all 3^6 FPRM expressions, for a 6-variable ternary function symmetric in k variables

Example 4.5 *Given a 10-variable ternary function, the symmetry vector* $S = [0011101111]$ *and polarity vector* $P = [1022011100]$*, the symmetry distribution vector is* $K(p) = [k_0 k_1 k_2] = [322]$*. For the assignment* $J = 0202211200$*, the assignment distribution vectors are* $K_0(p) = [201], K_1(p) = [011], K_2(p) = [101]$*. Fig. 3 illustrates from what bits of the vectors* $K_l(p)$ *they are formed.*

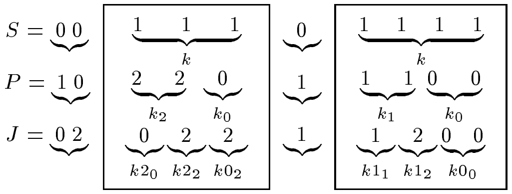

Figure 3. Structure of S, P and J

The assignments are called non-distinct if the sets of their assignment belong to the same $K_l(p)$. We only need to know N_c distinct coefficients of N_d distinct FPRM expressions to evaluate each of m^n forms within the FPRM family.

4.3 The product cost of the FRPRM expression

We evaluate below the number of products assigned to a distinct FPRM coefficient, i.e. the product cost of the coefficient, as well as the total product cost of the FPRM expression. Given a coefficient $r_p^{(j)}$, that corresponds to the assignment $J = j_1 j_2 \dots j_n$, $j_i \in \{0, 1, ..., m\}$, $i = 1, 2, ..., n$, with the assignment distribution vectors $K_l(p) =$

$[k_0, k_1, \ldots, k_{m-1}]$, we are interested in the number of product terms associated with the coefficient.

Theorem 4.4 *The product cost of the coefficient the p-polarity distinct FPRM expression for an m logic function symmetric in k variables, correspor the distinct assignment J with the set of the ass distribution vectors $K_l(p) = [kl_0, kl_1, ..., kl_{m-}$ $(0, 1, ..., m-1)$, is*

$$N_p(r_p^{(j)}) = \prod_{l=0}^{m-1} \binom{k_l}{kl_0, kl_1, ..., kl_{m-1}}$$

Example 4.6 *Given the assignment $J = 0202$ polarity distribution vector $K(p) = [k_0 k_1 k_2] = [3$ assignment distribution vectors $K_0(p) = [201], K$ $[011], K_2(p) = [101]$, the product cost of the coeffic is $N_p(r_p^{(j)}) = 12$. The coefficient $r_p^{(j)}$ is assigned to of 12 products: $(x_2^2 \widehat{\widehat{x}}{}^2_4 x_5^2 \widehat{x}_6 \widehat{x}_7 \widehat{x}_8^2 + x_2^2 \widehat{\widehat{x}}{}^2_4 \widehat{x}_6 \widehat{x}_7 \widehat{x}_8^2 x_9^2$ $x_2^2 \widehat{\widehat{x}}{}^2_3 \widehat{x}_6 \widehat{x}_7 \widehat{x}_8^2 x_{10}^2)$.*

Example 4.7 *Fig. 4 shows the distribution of t (number of product terms) to each distinct coeffi FPRM form of a 5-variable ternary function symm $2 \le k \le 5$ variables. For a 5-variable totally sy ternary function, the product cost is distributed bet distinct 0-polarity RM coefficients irregularly: the m (one product) is assigned to 10,15,20-th coefficie and the maximum (30 products) is assigned to 7,8, its coefficients. Information for partial symmetry obtained from the graph.*

Let a function f symmetric in k variables be rep by N_c distinct coefficients $r_p^{(j)}$ of the FPRM polarity p. Then, we can evaluate its cost as below.

Theorem 4.5 *The product cost of the p-polar expression of a logic function partially symmetr variables, is counted as*

$$N_p(f_p) = \sum_{j=0}^{N_c-1} r_p^{(j)} N_p(r_p^{(j)})$$

The proof follows from Theorem 4.4.

In the next section we estimate the percentage of realizations (realization of the smallest product cost the number of symmetric variables and polari variables, as well as the product costs for some ben functions.

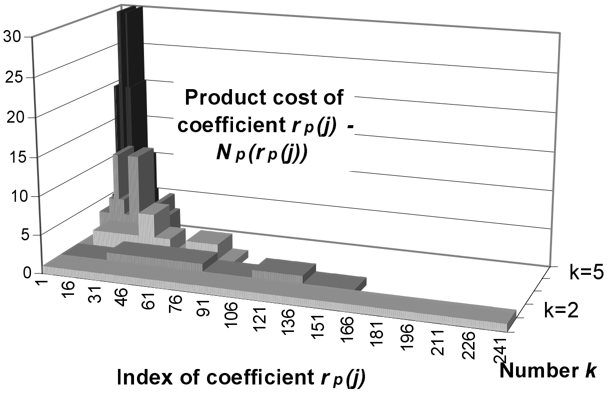

Figure 4. The product cost of j-th distinct coefficient $r_p^{(j)}$ of p-polarity RM expression of a 5-variable ternary function symmetric in k variables, $2 \le k \le 5$

5 Experimental results

Let us fix the number of symmetric variables and observe how the costs of FPRM coefficients depends on the polarity.

Fig. 5 illustrates the distribution of the product cost for each distinct coefficient in FPRM form of a 10-variable switching function symmetric in 6 variables. Note, that the symmetry is considered with respect to first six of 10 variables.

Now fix the polarity and the number of symmetric variables (9 of 10 variables) and observe, how the position of 0 in the symmetry vector (i.e. when non-symmetric variable is x_1, and then x_2, ..., and, finally, x_{10}) influences on the product cost of each coefficient in FPRM expression. The product cost of j-th distinct coefficient $r_0^{(j)}$ in FPRM expression of polarity $p = 0$ of a 10-variable switching function partially symmetric in 9 variables, with regard to the position of non-symmetric variable (marked by 0 in the symmetry vector $S = [1..101...1]$) is shown in Table 1.

We generated functions for small n and large k. We generate samples (200,000 of the whole class) for small k to obtain the approximated results. The goal of the experiment was to show the distribution of symmetric functions to the polarities that are optimally realize those functions.

Example 5.1 *Fig. 6 illustrates the distribution of the product cost of all FPRM expressions to the number k of symmetric variables of a ternary 5-variable function, $2 \le k \le 5$. It has been evaluated for 200,000 samples.*

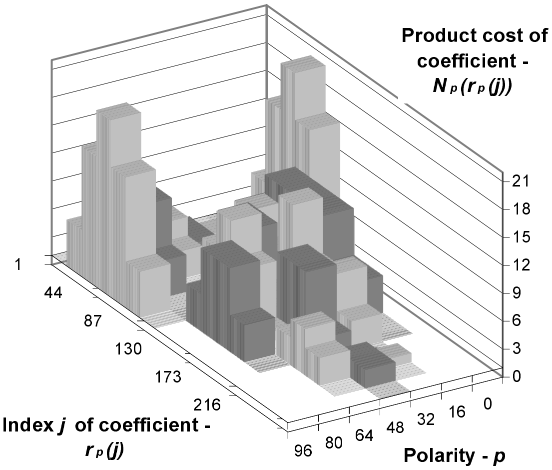

Figure 5. The product cost of j-th distinct coefficient $r_p^{(j)}$ of p-polarity RM expression of a 10-variable switching function symmetric in 6 variables

Table 1. The product cost of j-th distinct coefficient $r_p^{(j)}$ of 0-polarity RM expression of a 10-variable switching function symmetric in 9 variables, $S = [1..101...1]$

Index	Position of 0 in vector S									
	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	1	9	9	9	9	9	9	9	9	9
2	9	1	36	36	36	36	36	36	36	36
3	9	9	1	84	84	84	84	84	84	84
4	36	36	9	1	126	126	126	126	126	126
5	36	36	36	9	1	126	126	126	126	126
7	84	84	84	36	9	1	84	84	84	84
7	84	84	84	84	36	9	1	36	36	36
8	126	126	126	126	84	36	9	1	9	9
9	126	126	126	126	126	84	36	9	1	1
10	126	126	126	126	126	126	84	36	9	1
11	126	126	126	126	126	126	126	84	36	9
12	84	84	84	84	84	84	126	126	84	36
13	84	84	84	84	84	84	84	126	126	84
14	36	36	36	36	36	36	36	84	126	126
15	36	36	36	36	36	36	36	36	84	126
16	9	9	9	9	9	9	9	9	36	84
17	9	9	9	9	9	9	9	9	9	36
18	1	1	1	1	1	1	1	1	1	9
19	1	1	1	1	1	1	1	1	1	1

Table 2. Optimal polarity of FPRM expression of ternary functions

Test example			**Exact optimum**		
Name	**In**	**Sym.**	c	P/L	t
b11	3	x_2, x_3	122 (9)	1/1	0.00
b12	3	x_2, x_3	222	8/15	0.01
b13	3	x_2, x_3	022 (3)	2/2	0.01
b14	3	x_2, x_3	000 (9)	1/1	0.00
c171	5	$x_1, x_4; x_2, x_3$	00000 (6)	21/62	2.46
majority	5	$x_1, x_3 - x_5$	01220	38/145	2.45
xor5	5	$x_1 - x_5$	22222	2/5	2.45
cm138a1	6	$x_1, x_2; x_3 - x_6$	010000 (6)	65/353	46.12
cm138a2	6	$x_1, x_2, x_6; x_3 - x_5$	010001 (6)	65/353	46.00
cm138a3	6	$x_1, x_2, x_5; x_3, x_4, x_6$	010010 (6)	65/353	46.06
cm138a4	6	$x_1, x_2, x_5, x_6; x_3, x_4$	010011 (6)	65/353	46.07
cm138a5	6	$x_1, x_2, x_4; x_3, x_5, x_6$	010100 (6)	65/353	46.14
cm138a6	6	$x_1, x_2, x_4, x_6; x_3, x_5$	010101 (6)	65/353	46.25
cm138a7	6	$x_1, x_2, x_4, x_5; x_3, x_6$	010110 (6)	65/353	46.15
cm138a8	6	$x_1, x_2, x_4 - x_6$	010111 (6)	65/353	50.43

For ternary benchmark functions, we investigated how the number of optimal realizations are distributed with respect to various polarities. We generate the benchmarks from binary ones regards to interpreting gates AND as MIN, OR as MAX, and m-valued NOT of x as $(m-1)-x$. The results are given in Table 2. All experiments have been conducted on a Pentium 100Mhz with 48 MBytes of the main memory.[1]

For totally symmetric functions, we observe that for 3-valued functions, there are three ”best” polarities. These are 00...0, 11...1, and 22...2. The polarities that realize the fewest symmetric functions minimally have an equal or near equal number of 0’s, 1’s, and 2’s in the polarity code. The data suggests that, in general n-variable m-valued functions, the best polarity consists of all variables in the same inverted form. Since no single polarity realizes the best FPRM for the majority of functions, it is tempting to believe that, for each m, one should use m polarities, each consisting of all variables in the same inverted form. It is conjectured that many symmetric functions are realized minimally with this set of polarities.

6 Concluding remarks

This paper focuses on the FPRM expression for *partially symmetric* MVL functions. We derive the equations to estimate exactly the product of the FPRM expression of a given polarity and the number of variables of symmetry. First, we solve the problem in general mathematical formulation for arbitrary variables of symmetry, number of variables, polarity and the radix (Boolean and m-valued). Based on these general equations, we can evaluate any partially symmetric logic function. We use these equations to find an optimal polarity forms (with minimal product cost) for the given function, and show experimental results of this algorithm.

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[1] *The program has been written by Mr. P. Dziurzanski.*

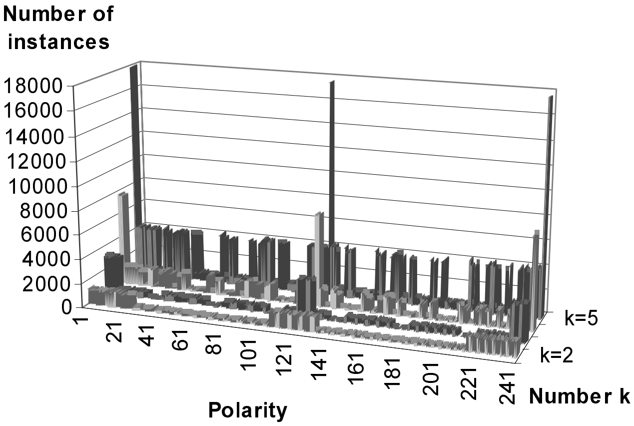

Figure 6. The distribution of optimal realizations of a 5-variable ternary function partially symmetric in $2 \leq k \leq 5$, to k and all fixed polarities

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