

## Two-Stage Exact Detection of Symmetries

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### Abstract

We propose an approach to exact detection of symmetries in multiple-valued logic (MVL) functions based on two-stage strategy. First, we reduce the search space by using information measures and then apply an exact method to find symmetry. The efficiency of the approach is evaluated by experimental study.

### 1 Introduction

We study *two-stage* approach to detect symmetries that includes reducing the search space and finding the exact solution in the remain space.

Referring to the previously obtained results, it should be noted the papers: [1, 4, 10] (properties of symmetries in MVL function), [3, 7] (detection of the symmetry in Fixed Polarity Reed-Muller (FPRM) form of switching functions), [6] (two-stage strategy for recognition of symmetry via ordered BDD).

The main problem of the two-stage exact strategy is to develop such a heuristic method that would efficiently reduce the search space without losses of possible symmetries, otherwise we cannot guarantee that all symmetries are included in reduced space. For example, the reduction of search space proposed in [6] is based on recognition of asymmetries in BDD, after that an exact algorithm is applied to the reduced space, so that to obtain a complete and exact solution.

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The main contribution of this paper is a new heuristic strategy for the reducing search space and, as a result, significant accelerating the symmetry detection. We prove that the entropy measure is ideally suitable for this heuristic strategy and satisfies some unique properties, in particular: (i) all possible symmetries are included in the rest of space and (ii) this method is applicable to both completely and incompletely specified function given either by cubes or decision diagrams.

For the first time, we introduce the entropy-based conditions to recognize symmetry in FPRM form of MVL functions.

### 2 Conditions to detect symmetry

**Definition 1** For any pair of variables  $\{x_i, x_j\}$  of an  $m$ -valued  $n$ -variables function  $f$ ,  $i, j \in (1, \dots, n)$ , its  $m^n$  values can be divided to  $(m^n - m)$  cofactors for  $x_i \neq x_j$  and  $m$  cofactors for  $x_i = x_j$ .

**Definition 2** [4] An  $m$ -valued function  $f$  of  $n$  variables is symmetric in variables  $x_i, x_j$  (denoted by  $x_i \sim x_j$ ) if  $f(x_i, x_j) = f(x_j, x_i)$  for all assignments of  $x_i$  and  $x_j$ .

There are  $(m^n - m)/2$  pairs of symmetric variables (with fixed value of remaining variables) for  $i \neq j$ . In particular case of switching functions, there are  $(2^n - 2)/2$  pairs  $f_{\bar{x}_i x_j} = f_{x_i \bar{x}_j}$ .

**Example 1** A 2-input ternary function given by the truth vector [012100201] has the same values for the following 3 pairs of variables of function:  $f(01) = f(10) = 1$ ,  $f(02) = f(20) = 2$  and  $f(12) = f(21) = 0$ . Hence, it is totally symmetric.

First, we formulate the conditions to recognize symmetries in a logic function, in terms of Logic Derivatives (LDRs). This result has a proper value and can be applied to recognize symmetry in FPRM (Fixed Polarity RM) form of logic function.

**Definition 3** The polarity vector of an FPRM expression of an  $n$ -variable  $m$ -valued function  $f$  is a set of polarities  $c_i$  of variables  $x_i$ ,  $[c_1 c_2 \dots c_n]$ , so that  $c = c_1 m^{n-1} + c_2 m^{n-2} + \dots + c_n m^0$ .

**Definition 4** The  $c$ -polarity Reed-Muller expression of an  $m$ -valued  $n$ -variable function  $f$  can be written as:

$$f_c = \sum_{l=0}^{m^n-1} f_c^{(l)} (x_1 + c_1)^{l_1} \dots (x_n + c_n)^{l_n}, \quad (1)$$

where sum is modulo  $m$ ,  $f_c^{(l)}$  is  $l$ -th coefficient whose value is either  $0, 1, \dots$  or  $m-1$ ,  $l_1, \dots, l_n$  is bit-wise representation of  $l$ ,  $l \in (0, m-1)$ .

**Definition 5** [9] The LDR of an  $m$ -valued function  $f$  with respect to the variable  $x_j$  with  $r_j$ -order cyclic inversion in the point  $x_i = x_j = a$  (other assignments are fixed) is as follows:

$$\left. \frac{\partial f}{\partial \hat{x}_j^{r_j}} \right|_{x_i=x_j=a} = \sum_{g=0}^{m-1} K_{m-r_j, m-a+g} f(x_i = a, x_j = g), \quad (2)$$

where  $a \in (0, \dots, m-1)$ ,  $K_{m-r_j, m-a+g}$  is the  $a$ -th element of the kernel matrix  $K_m^{(0)}$ ,  $m$  is a prime.

For example, the kernel matrices for  $m=2$  and  $m=3$  are as follows:

$$K_2^{(0)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad K_3^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$$

**Lemma 1** Let  $f$  be an  $m$ -valued function of  $n$  variables. Then  $x_i \sim x_j$  if any  $m-1$  of  $m$  conditions (each is the system of equations for  $h \in (1, \dots, m-1)$ )

$$\frac{\partial f(k, k)}{\partial \hat{x}_i^h} = \frac{\partial f(k, k)}{\partial \hat{x}_j^h}, \quad (3)$$

are satisfied for  $x_i = x_j$ , where  $f(k, k) = f(x_i = k, x_j = k)$ ,  $k \in (0, \dots, m-1)$ , and all operations are modulo  $m$ .

The proof-sketch is given in Appendix.

Note that the value  $f(k, k)$  is covered by  $m^{n-2}$  different values of function. Say,  $f(0, 0)$  for a 3-variable ternary function for  $\{x_1, x_2\}$  covers  $f(0, 0, 0)$ ,  $f(0, 0, 1)$  and  $f(0, 0, 2)$ .

In terms of FPRM coefficients  $f_c^{(l)}$ , condition (3) for a pair  $\{x_i, x_j\}$  can be re-written as bellow:

$$f_{kk}^{(J.h)} = f_{kk}^{(I.h)}, \quad (4)$$

where  $J = m^{n-j}$  and  $I = m^{n-i}$ .

**Table 1. FPRM coefficients and LDRs for example function  $f$**

Polarity	FPRM coefficients and LDRs					
	1	2	3	6	9	18
	001	002	010	020	100	200
0	000	2	0	0	0	0
1	001	2	1	0	0	0
2	002	2	2	0	0	0
12	110	0	0	0	1	0
13	111	0	0	1	0	1
14	112	0	0	1	0	1
24	220	0	0	2	2	2
25	221	0	0	0	0	0
26	222	0	0	0	0	0

**Example 2** To check whether a ternary function  $f = x_1^0 x_3^0 \vee x_2^0 x_3^0 \vee 2x_1^1 x_2^1$  is symmetric in  $\{x_1, x_2\}$ , it is enough to check two of three conditions (3) in terms of LDRs. Let us analyze these conditions for  $\{x_1, x_2\}$ ,  $h = 1$  and  $2$ ,  $k = 0$  and  $1$ :

$$\left\{ \begin{array}{l} \frac{\partial f(0,0)}{\partial \hat{x}_1} = \frac{\partial f(0,0)}{\partial \hat{x}_2} = \{0, 0, 0\}; \\ \frac{\partial f(0,0)}{\partial \hat{x}_1} = \frac{\partial f(0,0)}{\partial \hat{x}_2} = \{0, 0, 0\}; \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial f(1,1)}{\partial \hat{x}_1} = \frac{\partial f(1,1)}{\partial \hat{x}_2} = \{0, 1, 1\}; \\ \frac{\partial f(1,1)}{\partial \hat{x}_1} = \frac{\partial f(1,1)}{\partial \hat{x}_2} = \{1, 0, 0\}. \end{array} \right.$$

Here values in the brackets  $\{\}$  mean the values of LDRs for all 3 covered values, i.e.  $f(0, 0, 0)$ ,  $f(0, 0, 1)$  and  $f(0, 0, 2)$ . Note that we also could check other possible two of three conditions (3), for  $k = 1$  and  $2$  as well as  $k = 0$  and  $2$ .

For  $\{x_1, x_3\}$ , checking two of three possible conditions (3), implies:

$$\left\{ \begin{array}{l} \{0, 0, 0\} = \frac{\partial f(0,0)}{\partial \hat{x}_1} \neq \frac{\partial f(0,0)}{\partial \hat{x}_3} = \{2, 2, 2\}; \\ \{0, 0, 0\} = \frac{\partial f(0,0)}{\partial \hat{x}_1} \neq \frac{\partial f(0,0)}{\partial \hat{x}_3} = \{0, 1, 2\}; \end{array} \right.$$

$$\left\{ \begin{array}{l} \{0, 1, 1\} = \frac{\partial f(1,1)}{\partial \hat{x}_1} \neq \frac{\partial f(1,1)}{\partial \hat{x}_3} = \{0, 0, 0\}; \\ \{1, 0, 0\} = \frac{\partial f(1,1)}{\partial \hat{x}_1} \neq \frac{\partial f(1,1)}{\partial \hat{x}_3} = \{0, 0, 0\}. \end{array} \right.$$

i.e. no symmetry in  $\{x_1, x_3\}$ . So, the function is symmetric in  $\{x_1, x_2\}$ .

The selected values of LDRs that are the FPRM coefficients for 0-, 1-, 2-, 12-, 13-, 14- and 24-, 25-, 26- polarities, are shown in Table 1 (the first two columns contain the polarity codes). Note, that FPRM coefficients correspond to LDRs, for instance,  $f_{00}^{(9)} = \{0, 0, 0\} = \frac{\partial f(0,0)}{\partial \hat{x}_1}$ .

### 3 Entropy based strategy

LDRs give us the exact solution to detect symmetry in MVL functions. Based on our previous study [2, 11], we

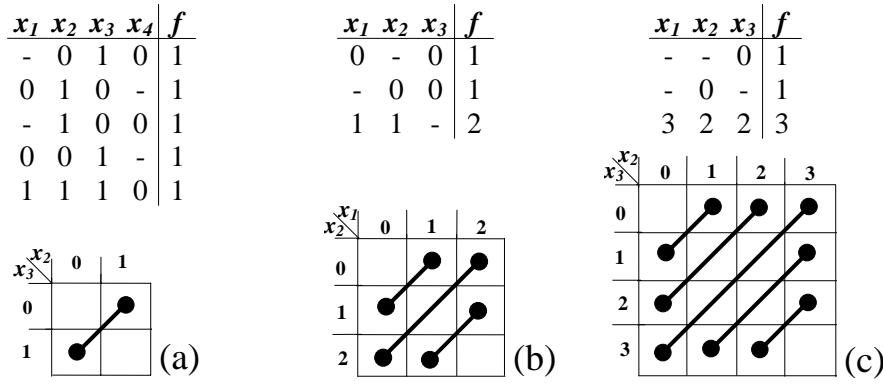


Figure 1. Symmetry in logic functions

will prove below that the entropy measure, namely, conditional entropy is suitable to reduce the search space at the first stage of symmetry recognition.

The conditional entropy of an  $m$ -valued function  $f$  with respect to variable  $x_i$  is given by [8]

$$H(f|x_i) = - \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} p_{|f=k, x_i=l} (\log_m p_{|f=k, x_i=l} - \log_m p_{|x_i=l}). \quad (5)$$

The probability  $p_{|f=k, x_i=l}$  is equal to the number of assignments  $f = k$  given  $x_i = l$ , divided by the total number of assignments, and  $p_{|x_i=l}$  is equal to the number of assignments  $x_i = j$  divided by the total number of assignments too.

Because the entropy measure is based on probabilistic estimations, we must answer the question: *can we lost some potentially symmetric variables?* We formulate, first, the necessary condition to recognize symmetry in MVL functions via entropy measure.

**Definition 6** The distribution of values  $f$  given  $x_i = a$ , denoted by  $\Delta_{x_i=a}$ , is a set  $\{\Delta_{f=0, x_i=a}, \Delta_{f=1, x_i=a}, \dots, \Delta_{f=m-1, x_i=a}\}$ , where  $\Delta_{f=b, x_i=a}$  is the number of cubes for which  $f = b$  given  $x_i = a$ ;  $a, b \in (0, \dots, m-1)$ . The distribution of fixed order is denoted as  $\Delta_{x_i=a}^*$ .

**Example 3** For the ternary function  $f = x_1^0 x_3^0 \vee x_2^0 x_3^0 \vee 2x_1^1 x_2^1$  (Fig. 1b), the distributions  $\Delta_{x_i=0}$ ,  $\Delta_{x_i=1}$ ,  $\Delta_{x_i=3}$  for each variable  $x_i$  are shown on Figure 2. For instance, for  $x_2 = 0$  function takes value 0 six times, value 1 three times, and value 3 zero times.

**Definition 7** The distribution of values of  $f$  given  $x_i$ , denoted by  $\Delta_{x_i}$ , is a set  $\{\Delta_{x_i=0}, \Delta_{x_i=1}, \dots, \Delta_{x_i=m-1}\}$ . The distribution of fixed order is denoted by  $\Delta_{x_i}^*$ .

**Definition 8**  $\alpha\{\Delta_{x_i}\}$  and  $\alpha\{\Delta_{x_i=j}\}$  mean permutation over the sets  $\Delta_{x_i}$  and  $\Delta_{x_i=j}$  respectively.

**Example 4** For the function  $f$  from Example 3, the distribution for  $x_1$  be  $\Delta_{x_1} = \{\Delta_{x_1=0}; \Delta_{x_1=1}; \Delta_{x_1=2}\} = \{\{6, 3, 0\}; \{5, 3, 1\}; \{8, 1, 0\}\}$ . Denote  $\{6, 3, 0\}$  as  $A$ ,  $\{5, 3, 1\}$  as  $B$ , and  $\{8, 1, 0\}$  as  $C$ . The permutation  $\alpha\{\Delta_{x_1}\}$  over this distribution is one of the sets:  $\{A; B; C\}$ ,  $\{A; C; B\}$ ,  $\{B; A; C\}$ ,  $\{B; C; A\}$ ,  $\{C; A; B\}$ ,  $\{C; B; A\}$ . In particular, the distribution for  $x_1 = 0$  be  $\Delta_{x_1=0} = \{6, 3, 0\}$ , so  $\alpha\{\Delta_{x_1=0}\} \in (\{6, 3, 0\}; \{6, 0, 3\}; \{3, 6, 0\}; \{3, 0, 6\}; \{0, 6, 3\}; \{0, 3, 6\})$ .

**Lemma 2** The cofactors of probabilities (the number of assignments) in the conditional entropy  $H(f|x_i)$ , are the elements within the permutation  $\alpha\{\Delta_{x_i}\}$ .

It is implied from that the entropy (5) includes all assignments  $f$  and  $x_i$ , and so  $\alpha\{\Delta_{x_i}\}$  includes them too.

**Example 5** For function from Example 3,  $H(f|x_1) = \frac{6}{27}(\log_3 2) + \frac{3}{27}(\log_3 1) + \frac{0}{27} + \frac{5}{27}(\log_3 \frac{5}{3}) + \frac{3}{27}(\log_3 1) + \frac{1}{27}(\log_3 \frac{1}{3}) + \frac{8}{27}(\log_3 \frac{8}{3}) + \frac{1}{27}(\log_3 \frac{1}{3}) + \frac{0}{27}$ . Here the bold numbers are included in  $\Delta_{x_1} = \{\{6, 3, 0\}, \{5, 3, 1\}, \{8, 1, 0\}\}$ .

The below Corollary establishes a relationships between  $\frac{\partial f}{\partial x_i}$  and  $\Delta_{x_i}$ .

**Corollary 1** If a function  $f$  is symmetric in  $\{x_i, x_j\}$ , i.e.  $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_j}$ , then there exist an equality between the distributions of fixed order for both variables:

$$\Delta_{x_i}^* = \Delta_{x_j}^*. \quad (6)$$

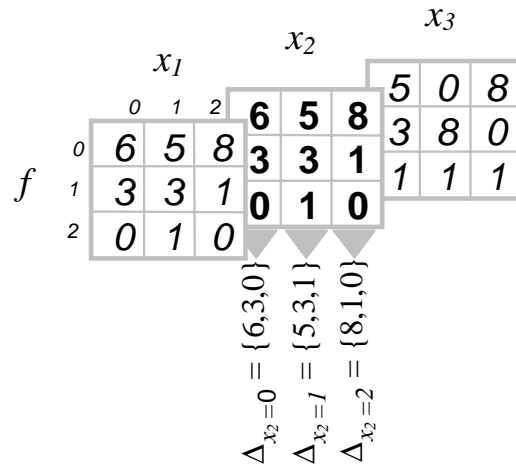
The proof-sketch is given in Appendix.

**Theorem 1** The necessary condition for a function  $f$  to be symmetric in  $\{x_i, x_j\}$  is the equality:

$$H(f|x_i) = H(f|x_j). \quad (7)$$

The proof follows from the definition of symmetry and Lemma 2, that implies

$$\alpha\{\Delta_{x_i}\} = \alpha\{\Delta_{x_j}\}. \quad (8)$$



**Figure 2. Distribution of variables of the function from Figure 1b**

**Corollary 2** Condition (7) is not sufficient to detect the symmetry  $x_i \sim x_j$ .

*Proof:* The distribution  $\Delta^* x_i$  of fixed order for LDR  $\frac{\partial f}{\partial x_i}$  is a subset of the permutation  $\alpha\{\Delta_{x_i}\}$ , that corresponds to  $H(f|x_i)$  as follows from Lemma 2:

$$\Delta_{x_i}^* \subset \alpha\{\Delta_{x_i}\}. \quad (9)$$

So, equality (7) can be true for symmetric and non-symmetric variables as well. Q.E.D

**Example 6** The switching function  $f = \bar{x}_2 x_3 \bar{x}_4 \vee \bar{x}_1 x_2 \bar{x}_3 \vee x_1 \bar{x}_2 \bar{x}_3 \vee \bar{x}_1 \bar{x}_2 x_3 \vee x_1 x_2 x_3 \bar{x}_4$  (Fig. 1a), and quaternary function  $f = x_3^0 \vee x_2^0 \vee 3x_1^3 x_2^2 x_3^2$  (Fig. 1c) are symmetric in  $\{x_1, x_2, x_3\}$  and  $\{x_2, x_3\}$  respectively. The entropies and distributions for these functions are as follows:

$$\begin{cases} H(f|x_1) = H(f|x_2) = H(f|x_3) = 0,977; \\ H(f|x_4) = 0,883; \end{cases} \downarrow \\ \begin{cases} \Delta_{x_1} = \{\{4,4\}; \{5,3\}\}; \\ \Delta_{x_2}^* = \Delta_{x_3}^* = \{\{5,3\}; \{4,4\}\}; \\ \Delta_{x_4=1} = \{\{3,5\}; \{2,6\}\}; \end{cases} \\ \begin{cases} H(f|x_2) = H(f|x_3) = 0,530; \\ H(f|x_1) = 0,343; \end{cases} \downarrow \\ \begin{cases} \Delta_{x_1} = \{\{9,7,0,0\}; \{9,7,0,0\}; \{9,7,0,0\}; \\ \{8,7,0,1\}\}; \\ \Delta_{x_2}^* = \Delta_{x_3}^* = \{\{0,16,0,0\}; \{12,4,0,0\}; \\ \{11,4,0,1\}; \{12,4,0,0\}\}; \end{cases}$$

Note, that  $\Delta_{x_2}^* = \Delta_{x_3}^*$  implies  $\frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3}$ , as shown in Figure 3.

**Corollary 3** If function  $f$  is symmetric in  $\{x_i, x_j\}$ , the conditional entropy for both variables is equal.

*Proof:* The symmetry  $x_i \sim x_j$  implies that  $\Delta_{x_i=a} = \Delta_{x_j=a}$ . It means that  $H(f|x_i) = H(f|x_j)$ . Q.E.D

Theorems (1) and (3) guarantee that we can perform the first stage to detect the symmetry without losing the symmetric variables. Notice that entropy based strategy to detect symmetry has been used in our recent study [2], in particular, in decision diagram based minimization [11].

## 4 Algorithm

Our algorithm (Fig. 4) combines the heuristic and exact stages. We first apply the heuristic stage to detect symmetry, i.e. we calculate entropy as necessary condition to find the potential symmetric pairs; then we find the exact solution that satisfies the necessary and sufficient condition.

## 5 Experiments

All our experiments have been performed on a Pentium III 450 MHz with 64 MB of memory. To verify the efficiency of the algorithm, we test it on about 30 single-output benchmarks<sup>1</sup>.

In Table 2 we give the results of the first stage of the algorithm, as well as the final results by exact strategy applied to the reduced search space. The first two columns provide information about the name of the function, primary inputs and the radix of logic. The third and the fourth columns show symmetric variables found at the first and second stages. We denote symmetry sets  $a$  of input variables

<sup>1</sup>The C source code for our package can be obtained from <http://www.wi.ps.pl/~atomaszewska.puc.dyda.wi/MVL/>.

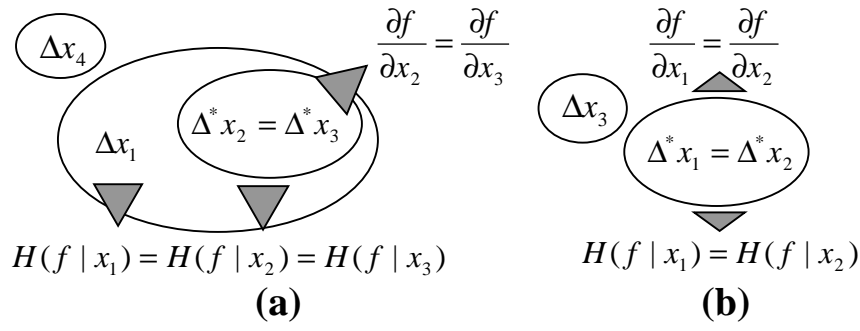


Figure 3. Dependence between entropy and LDR for functions from Figure 1a and 1b

```

(Stage1)
{
  for (forall xi in x1, x2, ..., xn) /*Counting propability*/
  for (forall a in 1, 2, ..., m)
    p(xi = a) ← prob[xi]
  for (forall xi in x1, x2, ..., xn) /*Counting joint probability*/
  for (forall a in 1, 2, ..., m)
  for (forall b in 1, 2, ..., m)
    p(f = a | xi = b) ← jointly[xi]
  for (forall xi in x1, x2, ..., xn) /*Counting conditional entropy*/
    H(f | xi) ← prob[xi, joint[xi]
  for (All pairs {xi, xj}) /*Finding the potential symmetric*/
  if (H(f | xi) = H(f | xj)) /*variables*/
    they will be returned
}
ExactDetection(Stage2)
{
  for (All pairs {xi, xj}) /*Exact finding the symmetric*/
  { /*variables*/
    for (forall m-1 conditions) where xi = xj
    if (conditions (3))
      then f is symmetric in {xi, xj}
  }
}

```

Figure 4. The sketch of the algorithm.

$b$  as  $a(b)$ . Per cent of reducing the search space  $R(\%)$  is showed in the fifth column; 0% means that evaluation of entropy did not discover any asymmetry, and the 100% means the first stage reports the lack of symmetry. The tests were specified for 1.000 cubes here.

We illustrate the behaviour of the combined strategy by Figure 5 on the benchmark  $ul\ 5$ , a 5-valued function of 60 variables symmetric in  $\{x_4, x_{49}, x_{54}\}$ ,  $\{x_6, x_{19}, x_{39}\}$ ,  $\{x_8, x_{42}\}$ ,  $\{x_9, x_{60}\}$ ,  $\{x_{20}, x_{57}\}$ ,  $\{x_{21}, x_{36}, x_{45}\}$ ,  $\{x_{23}, x_{40}\}$  and  $\{x_{41}, x_{43}, x_{44}\}$  (the axes  $x_i$  and  $x_j$  means the indexes of  $i$  and  $j$  of variables). We demonstrate how many potential solutions are generated by the heuristic algorithm (we call it fast prognosis), and only few of them are justified to be the final solution. At the first stage, the search space was reduced by 82% and gave a heuristic solution (small black dots). At the second stage,

our program processes only 18% of all pairs of variables (bigger grey dots). The heuristic stage is quite fast, save CPU and memory resources, enabling us to tackle problems for which the exact algorithm fails due to time or memory limits.

In Table 3, we compare the average run time of symmetry recognition (50 experiments for every test), for the programs working on whole and reduced search space, respectively. We assign don't cares in random way. The small tests (Experiment A) were completely specified, and the middle tests (Experiment B) were specified for 10.000 cubes. The first, the second, the fifth and the sixth columns contain information about the test functions. The third and the seventh columns show the run time of CPU, in seconds, for the proposed two-stage strategy. The fourth and the eighth columns show the run time of CPU for the naive strategy (algorithm working in the whole search space, without reduction). Notice, that the run time of the first stage was  $< 0,00$ . We observed that our algorithm performs 3 times faster for small benchmarks, and 61 times faster for middle tests.

In particular case, for switching functions, we demonstrate the effectiveness of our algorithm for some samples from LGSynth93 benchmark base (Table 4) compared with [7] and [5]. The number in the brackets means the number of symmetric variables in the group.

## 6 Concluding Remarks

The goal of this work was to establish the feasibility of two-stage technique for exact recognition of symmetry. We propose a heuristic method for the first stage, and proved that there are no losses of symmetry through reducing the search space. In addition, we suggest a method to recognize symmetry in FPRM form of MVL function, based on LDRs, that is a generalization of the well-known results for switching functions [5, 7].

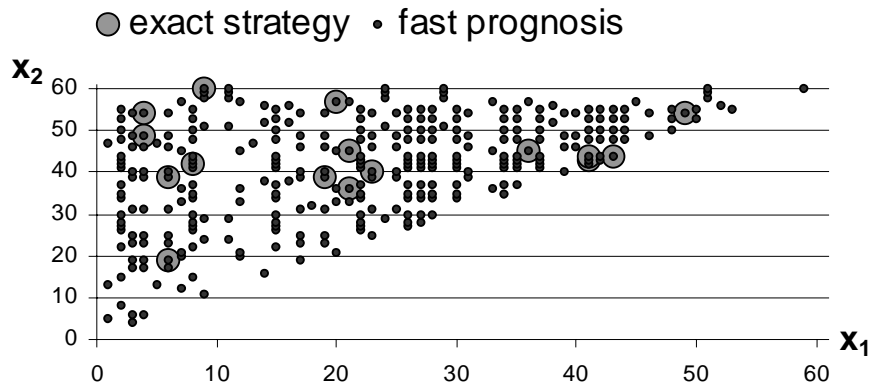


Figure 5. Example of heuristic and exact detection of the symmetry for 5-valued function  $u_{1.5}$  of 60 variables.

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## Appendix

*Proof-sketch of Lemma 1.* Assume  $x_i \sim x_j$ , i.e.  $f(x_i, x_j) = f(x_j, x_i)$ . By (5), when  $x_i = x_j = e$ , we get:  

$$\frac{\partial f(0,0)}{\partial x_i} = K_{m-1,m-e+1}f(x_i = 1, x_j = 0) + K_{m-1,m-e+2}f(x_i = 2, x_j = 0) + \dots + K_{m-1,2m-e-1}f(x_i = m-1, x_j = 0)$$

$$= K_{m-1,m-e+1}f(x_i = 0, x_j = 1) + K_{m-1,m-e+2}f(x_i = 0, x_j = 2) + \dots + K_{m-1,2m-e-1} \cdot f(x_i = 0, x_j = m-1) = \frac{\partial f(0,0)}{\partial x_j},$$
and so for other LDRs. Taking into account that [9]:

$$\begin{cases} K_{m-1,m-e+1} = K_{m-1,m-e+2} = \dots = K_{m-1,2m-e-1}, \\ K_{m-2,m-e+1} \neq K_{m-2,m-e+2} \neq \dots \neq K_{m-2,2m-e-1}, \\ \dots \\ K_{1,m-e+1} \neq K_{1,m-e+2} \neq \dots \neq K_{1,2m-e-1}, \end{cases}$$

we conclude that there is only one solution for the pair of variables  $\{x_i, x_j\}$ , and the same can be justified for the rest of conditions (3),  $i \in \{1, 2, \dots, m-1\}$ . So:

$$\begin{cases} f(1, 0) = f(0, 1) \\ f(2, 0) = f(0, 2) \\ \dots \\ f(m-1, m-2) = f(m-2, m-1) \end{cases} \quad Q.E.D$$

*Proof-sketch of Corollary 1:* The LDRs for a switching ( $m=2$ ) function  $f$  of  $n$  variables symmetric in  $\{x_i, x_j\}$  are:  $\frac{\partial f}{\partial x_i} = f(x_i = 0, x_j = 0) \oplus f(x_i = 1, x_j = 0)$ ,  $\frac{\partial f}{\partial x_j} = f(x_i = 0, x_j = 0) \oplus f(x_i = 0, x_j = 1)$ . If  $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_j}$

**Table 2. Estimation of effectiveness of reducing the search space**

Test		Symmetry		R(%)
Name	I/m	Heuristic	Exact	
mm3	5/3	{2,5}{3,4}	{2,5}{3,4}	80
c1a	6/3	{1,2,3,4,5}	{3,4}	33
c1b	6/3	{1,2,4,5}{3,6}	{1,4}{2,5}{3,6}	53
pal3	6/3	{1,2,3,4,5,6}	{1,6}{2,5}{3,4}	0
tic-tac-toe	9/3	{1,3,7,9}{2,4,6,8}	–	66
mm4	5/4	{2,5}{3,4}	{2,5}{3,4}	80
monks1te	6/4	{1,2,4}{3,6}	{1,2}{3,6}	73
monks2te	6/4	{1,2,4}{3,6}	{1,2}{3,6}	73
monks3te	6/4	{3,6}	{3,6}	93
balance	4/5	{1,2}{3,4}	{1,2}{3,4}	66
nursey	8/5	–	–	100
car	6/4	–	–	100
sim61	60/3	(41)(15)	2(2);4(3);(4);3(6); 2(7);(8)	40
u1_5	60/4	(2);(4);(5);2(8); (13);(19)	4(2);4(3)	82
u1_10	60/5	12(2);2(3);(4)	–	99
u2	60/5	{33,34,35,36}	{33,34,35,36}	99
u4	60/5	{25,26,27,28}	{25,26,27,28}	99
u5	60/5	{29,30,31,32}	{29,30,31,32}	99
c2a	11/7	{2,6}	–	98
c3a	11/7	{2,6}	–	98
c4a	11/7	{2,9}	–	98
zoo	16/7	{2,4}	–	99
trains	32/10	{3,11}{18,19,21} {25,30,32}{28,31}	{18,19}{25,30,32} {28,31}	98

then  $f(x_i = 1, x_j = 0) = f(x_i = 0, x_j = 1)$ , and so  $\{\Delta_{x_i=0}^*, \Delta_{x_i=1}^*\} = \{\Delta_{x_j=0}^*, \Delta_{x_j=1}^*\}$ . Let us analyze the LDRs for ternary function ( $m=3$ ) and 0-polarity:  $\frac{\partial f(0,0)}{\partial \hat{x}_i} = k_{2,3}f(x_i = 0, x_j = 0) + k_{2,2}f(x_i = 1, x_j = 0) + k_{2,1}f(x_i = 2, x_j = 0)$  and  $\frac{\partial f(0,0)}{\partial \hat{x}_j} = k_{2,3}f(x_i = 0, x_j = 0) + k_{2,2}f(x_i = 0, x_j = 1) + k_{2,1}f(x_i = 0, x_j = 2)$ ;  $\frac{\partial f(0,0)}{\partial \hat{x}_i} = k_{1,3}f(x_i = 0, x_j = 0) + k_{1,2}f(x_i = 1, x_j = 0) + k_{1,1}f(x_i = 2, x_j = 0)$  and  $\frac{\partial f(0,0)}{\partial \hat{x}_j} = k_{0,3}f(x_i = 0, x_j = 0) + k_{0,2}f(x_i = 0, x_j = 1) + k_{0,1}f(x_i = 0, x_j = 2)$ . According to (3), the symmetry in  $\{x_i, x_j\}$  is detected if both the equations  $\frac{\partial f(0,0)}{\partial \hat{x}_i} = \frac{\partial f(0,0)}{\partial \hat{x}_j}$  and  $\frac{\partial f(0,0)}{\partial \hat{x}_i} = \frac{\partial f(0,0)}{\partial \hat{x}_j}$  are satisfied. If so,  $\Delta_{x_i=0}^* = \Delta_{x_j=0}^*$ . Analogously,  $\Delta_{x_i=1}^* = \Delta_{x_j=1}^*$  and  $\Delta_{x_i=2}^* = \Delta_{x_j=2}^*$ . The generalization for arbitrary  $m$  implies (6). Q.E.D

**Table 3. Comparison of exact two-phase and single-phase strategies**

Test		CPU[s]		Test		CPU[s]	
Name	IN/k	Our	Naive	Name	IN/k	Our	Naive
<b>Exper. A (completely specified)</b>				<b>Exper. B (incompletely specified)</b>			
mm3	5/3	0,11	0,16	sim61	60/3	19,68	122,47
c1a	6/3	0,00	0,18	u1_5	60/4	14,50	352,42
c1b	6/3	0,00	0,20	u1_10	60/5	4,17	438,28
pal3	6/3	0,10	0,16	u2	60/5	0,82	535,82
tic-tac-toe	9/3	3,21	7,09	u4	60/5	0,82	435,22
mm4	5/4	0,38	0,51	u5	60/5	0,82	479,45
monks1te	6/4	0,00	0,66	c2a	11/7	0,00	4,56
monks3te	6/4	0,00	0,56	c3a	11/7	0,00	5,01
monks1tr	6/4	0,00	0,61	c4a	11/7	0,05	5,16
monks2tr	6/4	0,00	0,60	zoo	16/7	0,00	39,34
monks3tr	6/4	0,00	0,57	trains	32/10	4,61	389,34
<b>Total</b>		<b>3,80</b>	<b>11,30</b>	<b>Total</b>		<b>45,47</b>	<b>2807,07</b>

**Table 4. Comparison of three heuristic strategies**

Test	I/O	Heuristic [7]		Our heuristic	
		Symmetry	CPU[s]	Symmetry	CPU[s]
apex6_41	13/1	(10);(2)	0.691	(10);(2)	0,48
apex7_8	19/1	(8);(3);2(2)	4.917	(8);(3);2(2)	0,02
cm82_1	5/1	(3);2(2)	0.044	(3);2(2)	0,00
des_120	18/1	(4);7(2)	3.602	(13);(3);(17)	11,16
f51m_3	5/1	2(2)	0.113	(4)	0,11
x3_13	13/1	(10);(2)	0.715	(10);(2)	1,05
x4_66	10/1	(4);(3);(2)	0.275	(8);(4)	0,13
z4ml_1	7/1	(3);2(2)	0.113	(3);2(2)	0,00
<b>Heuristic [5]</b>					
9symml	9/1	(9)	–	(9)	0,02
cm138a	6/1	(2)	–	(6)	0,00
cm162a	14/1	(2)	–	(3);(2)	0,01
cm163a	16/1	(4)	–	(4);(2)	0,01
cm82a	5/1	(2)	–	(2);(3)	0,00
cmb	16/1	2(4)	–	2(4);(8)	0,01
count	35/1	(2)	–	(3)	0,07
frg1	28/1	(2)	–	2(2)	1,12
pm1	16/1	(3)	–	2(3);(2);(4)	0,00
term1	34/1	(2)	–	3(5);2(2);(4)	0,49
x2	10/1	(2)	–	(2);(3)	0,00
x3	135/1	(2)	–	5(2);(4);(3)	2,20
x4	94/1	(2)	–	(2)	0,82
z4ml	7/1	(2)	–	(3);2(2)	0,00